HW for 2019-09-23

(due: 2019-10-02)

You may (and should) talk about problems with each other and with me, providing attribution for any good ideas you might get. Your final write-up should be your own.

1: GECP and function approximation Suppose for a smooth function of two variables on the domain $[-1, 1]^2$, we seek an approximation as a sum of *separable functions*:

$$f_m(x, y) = \sum_{k=1}^{m} g_k(x)h_k(y).$$

One approach (due to Townsend and Trefethen) is the following greedy strategy: given $f_m(x, y)$ with $f_0 \equiv 0$, compute f_{m+1} as

$$\begin{aligned} e_m(x, y) &= f(x, y) - f_m(x, y) \\ x_m^*, y_m^* &= \operatorname{argmax}_{x,y} |e_m(x, y)| \\ g_{m+1}(x) &= e_m(x, y_m^*)/e_m(x_m^*, y_m^*) \\ h_{m+1}(y) &= e_m(x_m^*, y). \end{aligned}$$

Townsend and Trefethen refer to this as an infinite-dimensional version of Gaussian elimination with complete pivoting. Explain why this is a reasonable description, and implement the strategy yourself (you may restrict your attention to a finite mesh in x and y, if you wish). Illustrate the convergence of the maximum error over $[-1, 1]^2$ in the first twenty terms of approximating the squared exponential function $f(x, y) = \exp(-(x^2+y^2)/2)$ and the Ackley function

$$f(x, y) = 20 - 20 \exp\left(-0.2\sqrt{\frac{x^2 + y^2}{2}}\right) - \exp\left(\frac{1}{2}\cos(2\pi x) + \frac{1}{2}\cos(2\pi y)\right).$$

You may use the partial code templates provided in the Github repo (in Julia or Matlab), or you may write your own. Note that you will most likely want some mechanism to allow early termination if the error maximum of $|e_m|$ becomes too small (either in an absolute sense or relative to the maximum of $|f|$).

2: Convergence of iterative refinement In the second section of the lecture notes for Sep 23, we gave an informal discussion of the convergence of iterative refinement. In this problem, we nail down the analysis a little more. Consider the refinement iteration

$$x_{k+1} = x_k + \hat{A}_k^{-1}(b - Ax_k + g_k) + h_k$$

where $\hat{A}_k = A + E_k$ and g_k and h_k denote the rounding errors associated with computing the residual $b - Ax_k$ and doing the final subtraction in floating point arithmetic. Assuming $\|E_k\| \leq \eta$, $\|g_k\| \leq \gamma$, and $\|h_k\| \leq \nu$ for all k, and assuming that $\|A^{-1}\| < \eta^{-1}/2$, use a geometric series to bound the error at step k in terms of the initial error and the rounding errors introduced at each step. Because of the presence of g_k and h_k, the error will *not* converge to zero; what is an asymptotic bound on the norm of the error as $k \rightarrow \infty$? What does your analysis suggest about the number of steps of iterative refinement that one should take?

Hint: Observe that in general if one has non-negative quantities β_k satisfying

$$\beta_{k+1} \leq \rho\beta_k + \delta$$

then by recurrence

$$\beta_k \leq \rho^k\beta_0 + \frac{1-\rho^k}{1-\rho}\delta,$$

and by a geometric series bound, for $0 \leq \rho < 1$,

$$\beta_k \leq \rho^k\beta_0 + \frac{\delta}{1-\rho}$$

3: Condititioning of the complement Suppose A is symmetric and positive definite. Show that if S is a Schur complement that appears during (unpivoted) factorization of A, then $\kappa_2(S) \leq \kappa_2(A)$. Show by example that this statement may not hold if A is not positive definite.