CS 6210 Assignment 6   Due: 12/2/15 (Wed) at 11pm

Scoring for each problem is on a 0-to-5 scale (5 = complete success, 4 = overlooked a small detail, 3 = good job on half the problem, 2 = OK job on half the problem, 1 = germ of a relevant solution idea, 0 = missed the point of the problem.) Independent of this, one point will be deducted for insufficiently commented code. Test code and related material are posted on the course website `http://www.cs.cornell.edu/courses/cs6210/2015fa/`. All solution M-Files must be submitted through the CMS system. You are allowed to discuss *background* issues with other students, but the codes you submit must be your own.

Topics: Unsymmetric Eigenproblem, Real Schur Decomposition, Invariant Subspaces

1 Eigenvectors of ST

Complete the following function so that it performs as specified:

```
  function x = EvecProd(S,T,k)
% S and T are nxn upper triangular matrices and k is an integer with 1<=k<=n.
% Assume that the eigenvalues of ST are distinct, i.e., S(1,1)T(1,1),...,S(n,n)T(n,n)
% are distinct.
% x is a unit 2-norm column vector so that S*T*x = lambda*x  where lambda = S(k,k)T(k,)
```

A successful implementation will be $O(n^2)$ so this rules out any method that explicitly computes the upper triangular matrix product ST. Fortunately, it is possible to solve n-by-n linear systems of the form

$$(\text{Upper Triangular Matrix}) \times (\text{Upper Triangular Matrix}) \; - \; \mu I_n \; = \; \text{rhs}$$

with $O(n^2)$ work. (See GVL4 Problem 3.1.5.). This is relevant for if

$$ST \;=\; U \;=\; \begin{bmatrix} U_{11} & w & U_{13} \\ 0 & \lambda & v^T \\ 0 & 0 & U_{33} \end{bmatrix}$$

has square diagonal blocks and λ is not an eigenvalue of U_{11}, then

$$U \begin{bmatrix} (U_{11} - \lambda I)^{-1} w \\ -1 \\ 0 \end{bmatrix} \;=\; \lambda \begin{bmatrix} (U_{11} - \lambda I)^{-1} w \\ -1 \\ 0 \end{bmatrix}.$$

Submit `EvecProd` to CMS.

2 Best 2-by-2

Suppose $A \in \mathbb{R}^{n\times n}$ and that $Q_1 \in \mathbb{R}^{n\times 2}$ has orthonormal columns. If ran(Q) is an invariant subspace for A, then there exists $B \in \mathbb{R}^{2\times 2}$ so that $AQ = QB$. Each eigenvalue of B is an eigenvalue of A

If ran(X) is an approximate invariant subspace for A, then we wish to determine B so that $\|AX = XB\|_F$ is minimized. In this case the eigenvalues of B can regarded as approximate eigenvalues of A. Implement the following function so that it performs as specified

```
 function B = Best2by2(A,X)
% A is nxn, X is nx2
% B is a 2x2 matrix that minimizes norm(AX-XB,'fro').
```

Submit `Best2by2` to CMS.

3 Best n-by-2

If the eigenvalues of $B \in \mathbb{R}^{2\times 2}$ are also eigenvalues of $A \in \mathbb{R}^{n\times n}$, then there exists a unit Frobenius norm matrix $X \in \mathbb{R}^{n\times 2}$ such that $AX = XB$. If the eigenvalues of B are almost eigenvalues of A, then there should be a matrix $X \in \mathbb{R}^{n\times 2}$ such that $\| AX - XB \|_2$ is small subject to the constraint that $\| X \|_F = 1$. Implement the following function so that it performs as specified

```
  function X = BestNby2(A,B)
 % A is nxn, B is 2x2
 % X is an nx2 matrix that minimizes norm(AX-XB,'fro') subject to norm(X,'fro') = 1.
```

Submit `BestNby2` to CMS.

4 Structured Real Schur Decomposition

Read up on `schur` and `ordschur` and then implement the following function:

```
   function [U,T] = specialSchur(X,Y,order)
 % X and Y are n-by-r matrices with r<n/2.
 % U is orthogonal and U'*(X*Y')*U = T is the real Schur decomposition of XY'.
 % If order = 'ascend' then the eigenvalues along the block diagonal of T are arranged
 %       in order of increasing real part.
 % If order = 'descend' then the eigenvalues along the block diagonal of T are arranged
 %       in order of decreasing real part.
 %
```

A good way to focus on efficiency issues is to consider the case $r << n$. Submit `specialSchur` to CMS.