CS 6210: Take-Home Final (Revised)

Due: Friday, December 10, 2010 (In Class or in Upson 5153 by 4pm)

1. Each of problem is worth 10 points and involves writing a single MATLAB function. (The functions can have subfunctions.)

2. The three solution functions should be sent as separate attachments in a *single* email to `cv@cs.cornell.edu`. No later than 4pm, Friday, December 10, 2010.

3. To facilitate grading, hard copies of the function listings should also be submitted in 5153 Upson. No later than 4pm, Friday, December 10, 2010.

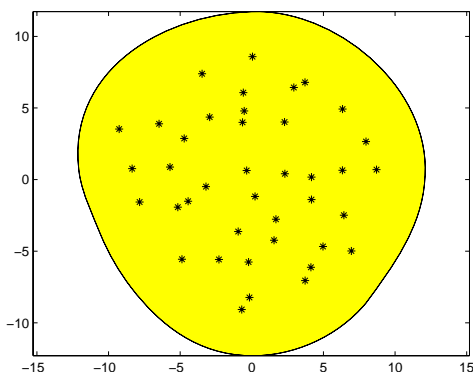4. You are not allowed to discuss the exam questions with anybody except me.

P1. (Field of Values)

If $A \in \mathbb{C}^{n \times n}$, then its *field of values* is the set

$$W(A) \; = \; \left\{ x^H A x : \| x \|_2 \leq 1 \right\}.$$

Note that $\lambda(A)$ is a subset of $W(A)$ and that $0 \in W(A)$. If A is Hermitian [1], then $W(A) = [\lambda_{min}(A), \lambda_{max}(A)]$.

In this problem you are to write a function `FieldOfValues(A)` that displays $W(A)$ for an arbitrary complex A together with its eigenvalues:

A key properties of $W(A)$ include

1. $W(A)$ is convex.

2. If $c \in \mathbb{C}$ then $W(cA) = cW(A) \equiv \{ cz^H A z : \| z \| \leq 1 \}$

and this permits us to easily compute points on its boundary as we now show. Suppose $0 \leq \theta \leq 2\pi$ and define B_θ by

$$B_\theta = \frac{(e^{-i\theta} A) + (e^{-i\theta} A)^H}{2}.$$

Since B_θ is Hermitian it has real eigenvalues

$$\lambda_n = \lambda_n(B_\theta) \leq \cdots \leq \lambda_1(B_\theta) \; = \; \lambda_1$$

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[1] If $A \in \mathbb{R}^{n \times n}$ is Hermitian then $A^H = A$ where A^H is the conjugate transpose of A. Note that in MATLAB, `A'` produces A^H. Everything about the symmetric eigenproblem carries over to the Hermitian eigenproblem. If A is Hermitian, then it has a Schur decomposition $Q^H A Q = \text{diag}(\lambda_i)$ where the λ_i are real and Q is unitary, i.e., $Q^H Q = I$. The perturbation theory carries over (e.g. the Wielandt-Hoffman theorem) and all the algorithms (e.g., Jacobi's method).

and an orthonormal basis of eigenvectors $B_\theta z_i = \lambda_i(B_\theta)z_i$, $i = 1{:}n$. We always have $\lambda_n \le z^H B_\theta z \le \lambda_1$. The upper is attained by setting $z = z_1$ and the lower bound is attained by setting $z = z_n$. Note that

$$z^H B_\theta z \;=\; \mathrm{real}(z^H \left(e^{-i\theta}A\right)z)$$

and so for unit vectors z we have

$$\lambda_n(B_\theta) \;\le\; \mathrm{real}(z^H \left(e^{-i\theta}A\right)z) \;\le\; \lambda_1(B_\theta)$$

If $z = z_1$, then the upper bound is realized and $z^H(e^{-i\theta}A)z$ is on the boundary of $W(e^{-i\theta}A) = e^{-i\theta}W(A)$. Thus, $z_1^H A z_1$ is on the boundary of $W(A)$. Similarly, $z_n^H A z_n$ is on the boundary of $W(A)$.

Your implementation of `FieldOfValues(A)` should proceed by computing the $W(A)$ boundary points $z_1^H A z_1$ and $z_n^H A z_n$ for a sufficiently large number of θ and then use the results to graphically display $W(A)$. Remember that $W(A)$ is convex and includes 0.

For the eigenvector computations you must implement a Hermitian version of Jacobi's method. Exploit the fact that if τ and μ are close and $Q_\mu^T B_\mu Q_\mu = D_\mu$ is a Schur decomposition, then $Q_\mu^T B_\tau Q_\mu$ is almost diagonal. For a graphical display, you do not need 10^{-16} accuracy, so choose the number of θ values and the Jacobi termination criteria accordingly and include comments in your code that justify your choices. You may assume that $n \le 100$. Your $W(A)$ display should more or less look like the above figure. Note that if z is a complex vector then `plot(real(z),imag(z),'*')` displays the complex components and `fill(real(z),imag(z),'y')` produces a filled polygon defined by `z`. You can use `eig` to compute A's eigenvalues.

P2. (Leveling Networks)

Suppose $(x_1, y_1), \ldots, (x_n, y_n)$ are distinct points on the Cornell campus. Our goal is to estimate the elevation h_i of point (x_i, y_i) given that (a) a surveyor handed us a matrix $E = (e_{ij}) \in \mathbb{R}^{n \times n}$ in sparse format where e_{ij} is a noisy estimate of $h_j - h_i$ and that (b) h_1 is given. Let S be the set of (i, j) pairs with the property that if $(i, j) \in S$, then the surveyor estimated $h_j - h_i$. This set can be deduced from the sparse format matrix E. Assume that if $(i, j) \in S$, then $i \ne j$.

Write a MATLAB function `h = Elevations(x,y,E,h1)` that returns a column n-vector h that minimizes

$$\phi(h_1, h_2, \ldots, h_n) \;=\; \sum_{(i,j)\in S} |h_j - h_i - e_{ij}|^2/d_{ij}^2$$

where h_1 is specified and

$$d_{ij} \;=\; \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}.$$

(Note that measurements between close points are emphasized in the summation.) This is a min $\| Ax - b \|_2$ problem which you may assume has full rank. Store A in sparse format and use $\backslash$. The value of the first component of the output vector `h` should the value of `h1`.

P3. (A Structured Block Tridiagonal Solver)

Suppose A symmetric, positive definite, and block tridiagonal:

$$A = \begin{bmatrix} T_1 & C_1^T & \cdots & & 0 \\ C_1 & T_2 & & & \vdots \\ \vdots & \ddots & \ddots & & C_{p-1}^T \\ 0 & \cdots & & C_{p-1} & T_p \end{bmatrix} \qquad T_i, C_i \in \mathbb{R}^{q \times q}$$

Assume that $T_1, \ldots, T_p$ are square, large and sparse and that $C_i = f_i g_i^T$ for $i = 1{:}p-1$ where $f_i, g_i \in \mathbb{R}^q$. Write a function `x = BlockTri(T,f,g,b)` that returns the solution to $Ax = b$ where

- T is a length p cell array with the property that the value of $\mathsf{T\{i\}}$ is T_i, a q-by-q matrix in sparse format.

- $n = pq$, b is a column n-vector, and x is a column n-vector such that $Ax = b$.

- f is a length $p-1$ cell array with the property that $\mathsf{f\{i\}}$ houses the column q-vector f_i, $i = 1{:}p-1$.

- g is a length $p-1$ cell array with the property that $\mathsf{g\{i\}}$ houses the column q-vector g_i, $i = 1{:}p-1$.

Make effective use of pcg and $\mathsf{cholinc}$. These methods have various options and your code should be structured so that it is easy for me to see (and possibly) vary your choices.

REMINDER: IF YOUR SOLUTION CODE INVOLVES A DESIGN CHOICE, IT IS IMPORTANT THAT YOU INCLUDE COMMENTS THAT JUSTIFY THAT CHOICE AND THAT YOU CODE IS CLEAR ENOUGH SO THAT I CAN MODIFY THOSE CHOICES (IF NECESSARY) FOR GRADING.