CS 6210: Challenge 1 Notes

What follows points to how the product can be computed in $O(nk)$ flops.

Suppose $n = 5$ and that

$$p = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \end{bmatrix} \qquad q = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \end{bmatrix} \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \qquad d = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \end{bmatrix}$$

If

$$L = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ q_2p_1 & 0 & 0 & 0 & 0 \\ q_3p_1 & q_3p_2 & 0 & 0 & 0 \\ q_4p_1 & q_4p_2 & q_4p_3 & 0 & 0 \\ q_5p_1 & q_5p_2 & q_5p_3 & q_5p_4 & 0 \end{bmatrix}$$

and $u = p.*x$, then

$$\begin{bmatrix} \tilde{u}_1 \\ \tilde{u}_2 \\ \tilde{u}_3 \\ \tilde{u}_4 \\ \tilde{u}_5 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ q_2p_1 & 0 & 0 & 0 & 0 \\ q_3p_1 & q_3p_2 & 0 & 0 & 0 \\ q_4p_1 & q_4p_2 & q_4p_3 & 0 & 0 \\ q_5p_1 & q_5p_2 & q_5p_3 & q_5p_4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ q_2u_1 \\ q_3(u_1+u_2) \\ q_4(u_1+u_2+u_3) \\ q_5(u_1+u_2+u_3+u_4) \end{bmatrix}$$

This matrix-vector product can be carried out as follows:

> $u = p.*x$; $\tilde{u}_1 = 0$; $s = 0$
> **for** $j = 2{:}n$
> $s = s + u_{j-1}$
> $\tilde{u}_j = q_j s$
> **end**

Approximately $2n$ flops are required. Suppose we have a function `utilde = LowerProd(p,q,x)` that carries this out and that $P, Q \in \mathbb{R}^{n\times k}$. Since

$$\text{tril}(QP^T, -1)x \;=\; \sum_{j=1}^{k} \text{tril}(Q(:,j)P(:,j)^T, -1)x$$

we see that we can evaluate $\text{tril}(QP^T, -1)x$ in $2kn$ flops.

Clearly, we can design an analogous function `utilde = UpperProd(p,q,x)` and use it to evaluate $\text{triu}(PQ^T, 1)x$ in $2kn$ flops. Since $\text{diag}(d)x = d.*x$ costs n flops we see that

$$y \;=\; Ax \;=\text{tril}(QP^T,-1)x \;+\; \text{diag}(d)x \;+\; \text{triu}(PQ^T,1)x$$

can be evaluated in $(2kn + n + n + n + 2kn) = (4k+3)n$ flops.