Assignment 3

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Exercise 1. **Set** has a cool property. Suppose you have a commuting square

$$\begin{array}{ccc} A & \overset{e}{\twoheadrightarrow} & B \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ C & \underset{m}{\hookrightarrow} & D \end{array}$$

where e is a surjection and m is an injection. Then there exists a (unique) diagonal $d : B \to C$ such that the following commutes:

$$\begin{array}{ccc} A & \overset{e}{\twoheadrightarrow} & B \\ {\scriptstyle f}\downarrow & \overset{d}{\swarrow} & \downarrow{\scriptstyle g} \\ C & \underset{m}{\hookrightarrow} & D \end{array}$$

This function is given by $d(b) = f(a)$ for any $a \in A$ such that $e(a) = b$. The reason this function is well-defined is two-fold. First, because e is surjective, for every input $b \in B$ there necessarily exists some $a \in B$ such that $e(a) = b$, thereby making d total. Second, for any two a and a' in A such that $e(a) = e(a') = b$, $f(a)$ necessarily equals $f(a')$, making d determined. The reason is that m is injective, so $f(a) = f(a')$ holds if $m(f(a)) = m(f(a'))$ holds, and the latter is equivalent to $g(e(a)) = g(e(a'))$ because the square commutes, which is then equivalent to $g(b) = g(b)$ because of the assumed equalities, which clearly holds by reflexivity.

From a categorical perspective, this proof is actually just applying the fact that every surjection is a regular epimorphism, every injection is a monomorphism, and every category has (unique) (RegEpi,Mono)-diagonalizations. This last property of a given category **C** means that, given any commuting square as above such that e is a regular epimorphism and m is a monomorphism in **C**, then there exists a (unique) commuting diagonal d as above. Prove that every category has (RegEpi,Mono)-diagonalizations.

Alternatively, prove that every category has (Epi,RegMono)-diagonalizations: given any commuting square as above such that e is an epimorphism and m is a regular monomorphism in **C**, then there exists a (unique) commuting diagonal d as above.

Definition. A kernel pair of a morphism $f : A \to B$ is a parallel pair of morphisms $R \underset{k_2}{\overset{k_1}{\rightrightarrows}} A$ such that $k_1 \,; f = k_2 \,; f$ with the property that for any other parallel pair $X \underset{g_2}{\overset{g_1}{\rightrightarrows}} A$ satisfying $g_1 \,; f = g_2 \,; f$ there exists a unique morphism $u : X \to R$ satisfying $g_1 = u \,; k_1$ and $g_2 = u \,; k_2$.

Exercise 2. Suppose that, in some category **C**, every morphism has a kernel pair, every kernel pair has a coequalizer, and every regular epimorphism is a retraction. Prove that **C** has (RegEpi,Mono)-factorizations, meaning for every morphism $f : A \to B$ there exists a regular epimorphism $e : A \twoheadrightarrow C$ and monomorphism $m : C \hookrightarrow B$ such that f equals $e \,; m$. Note: unique (RegEpi,Mono)-diagonalizations can be used to prove that (RegEpi,Mono)-factorizations are always unique up to isomorphism.