# Practice 10

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Definition (Semiring). A semiring is a set A along with nullary operators $0 \in A$ and $1 \in A$ and binary operators $+ : A \times A \to A$ and $* : A \times A \to A$ such that $\langle A, 0, + \rangle$ forms a commutative (meaning $\forall a, a' \in A.\ a + a' = a' + a$) monoid, $\langle A, 1, * \rangle$ forms a (not necessarily commutative) monoid, and $*$ distributes *over* 0 and $+$, meaning the following properties hold:

- $\forall a \in A.\ a * 0 = 0$

- $\forall a, a_1, a_2 \in A.\ a * (a_1 + a_2) = (a * a_1) + (a * a_2)$

- $\forall a \in A.\ 0 * a = 0$

- $\forall a, a_1, a_2 \in A.\ (a_1 + a_2) * a = (a_1 * a) + (a_2 * a)$

Example. Addition and multiplication on each of the sets $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{R}$ are each examples of semirings. Addition and multiplication on $n \times n$ matrices is a semiring for any natural number n. Union and intersection on $\mathbb{P}X$ is a semiring for any set X. Maximum (as $+$) and addition (as $*$) on $\mathbb{N}$ is a semiring.

Remark. Given a multiplication of additions, distributivity of multiplication over addition implies it can be turned into an equivalent addition of multiplications. That is, given a multiplication-of-additions expression like $(a_1 + a_2 + a_3) * (a_4) * (a_5 + a_6)$, we can distribute the multiplications over the additions to get an equivalent addition-of-multiplications expression $(a_1 * a_4 * a_5) + (a_1 * a_4 * a_6) + (a_2 * a_4 * a_5) + (a_2 * a_4 * a_6) + (a_3 * a_4 * a_5) + (a_3 * a_4 * a_6)$. Note that we can also get $(a_1 * a_4 * a_5) + (a_2 * a_4 * a_5) + (a_3 * a_4 * a_5) + (a_1 * a_4 * a_6) + (a_2 * a_4 * a_6) + (a_3 * a_4 * a_6)$, which has the same multiplications but the order of additions is slightly different. So we can describe a multiplication of additions as a list of multisets/bags, and additions of multiplications as a multiset/bag of lists, and distributivity provides a way to transform a list of multisets/bags into a multiset/bag of lists. In fact, this distributivity describes a natural tranformation $\delta : \mathbb{M} \,;\, \mathbb{L} \Rightarrow \mathbb{L} \,;\, \mathbb{M}$ that distributes lists *over* multisets/bags. This behavior inspires the following concept.

Definition. A distributive law of a monad $\langle M_1, \eta_1, \mu_1 \rangle$ over a monad $\langle M_2, \eta_2, \mu_2 \rangle$ both on the same category $\mathbf{C}$ is a natural transformation $\delta : M_2 \,;\, M_1 \Rightarrow M_1 \,;\, M_2$ satisfying the following four equalities:
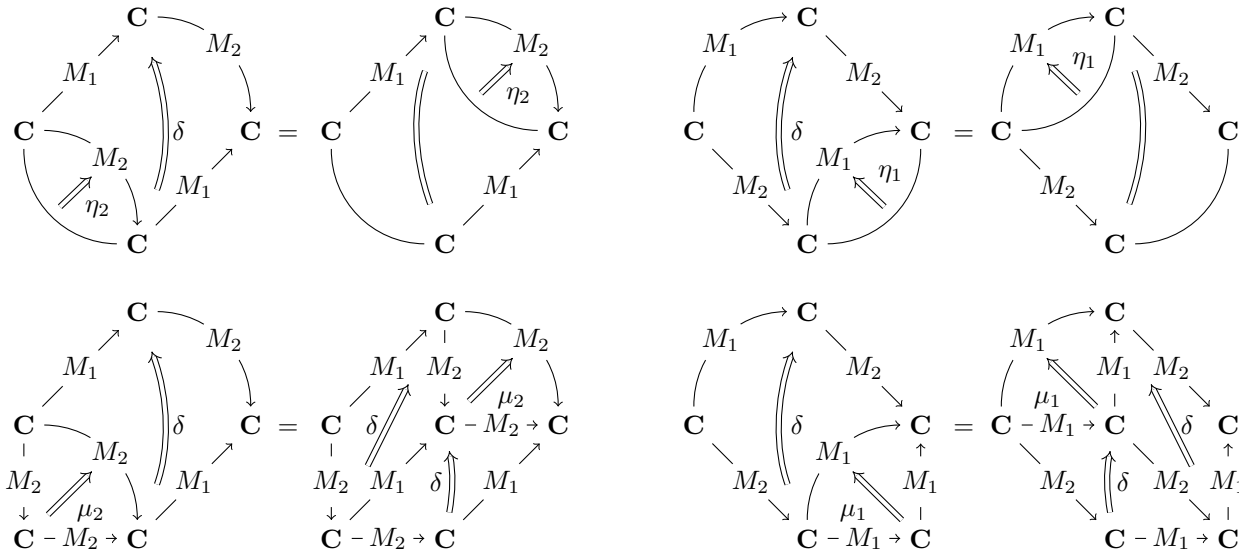
Remark. There is a (non-identity) natural transformation from $\mathbb{L} \,;\, \mathbb{L}$ to $\mathbb{L} \,;\, \mathbb{L}$ that corresponds to distributing a multiplication of (a list of) additions (of lists) into an addition of (a list of) multiplications (of lists). However, it fails to be a distributive law because, as shown before, different ways of distributing multiplications over additions can lead to different orderings of the multiplications in the resulting addition. This is why generalizations of semirings that remove the commutativity requirement of addition always also remove one of the distributivity axioms.

Exercise 1. Given a distributivity law δ of a monad $\langle M_1, \eta_1, \mu_1 \rangle$ over another monad $\langle M_2, \eta_2, \mu_2 \rangle$ both on some category $\mathbf{C}$, the composition of functors $M_1 \,;M_2$ has a unit η and join μ such that $\langle M_1 \,; M_2, \eta, \mu \rangle$ is a monad on C. Define η and μ such that they satisfy the monad laws, but only explicitly prove either the left- or right-identity law of the monad. You may use cell diagrams, string diagrams, or traditional algebraic formulae to formulate your definitions and explain your proof.

Remark. A semiring is a monad algebra of the monad $\mathbb{L}\,;\mathbb{M}$ built from the aforementioned distributive law of $\mathbb{L}$ over $\mathbb{M}$. Note that multiplication in a semiring is a monad algebra of $\mathbb{L}$ and addition in a semiring is a monad algebra of $\mathbb{M}$. Thus a semiring is simply a set equipped with both a monad algebra of $\mathbb{L}$ and a monad algebra of $\mathbb{M}$ satisfying an additional property as to how those two algebraic structures interact with each other, in particular that multiplications distribute over additions.

Exercise 2. Given monads $\langle M_1, \eta_1, \mu_1 \rangle$ and $\langle M_2, \eta_2, \mu_2 \rangle$ on a category $\mathbf{C}$, there is a sink $\mathbf{C}^{M_1} \xrightarrow{U_1} \mathbf{C} \xleftarrow{U_2} \mathbf{C}^{M_2}$ in $\mathbf{Cat}$. This sink has a pullback, often denoted somewhat ambiguously by $\mathbf{C}^{M_1} \times_{\mathbf{C}} \mathbf{C}^{M_2}$, because $\mathbf{Cat}$ is complete (meaning $\mathbf{Cat}$ has all small limits). An object of this pullback is an object $A \in \mathbf{C}$ along with morphisms $a_1 : M_1 A \to A$ and $a_2 : M_2 A \to A$ that satisfy the requirements of monad algebras of $\langle M_1, \eta_1, \mu_1 \rangle$ and $\langle M_2, \eta_2, \mu_2 \rangle$ respectively. Given a distributive law δ of $\langle M_1, \eta_1, \mu_1 \rangle$ over $\langle M_2, \eta_2, \mu_2 \rangle$ and the resulting monad $\langle M_1 \,; M_2, \eta, \mu \rangle$, the category $\mathbf{C}^{M_1 \,; M_2}$ is actually a full subcategory of $\mathbf{C}^{M_1} \times_{\mathbf{C}} \mathbf{C}^{M_2}$. Define the property an object of $\mathbf{C}^{M_1} \times_{\mathbf{C}} \mathbf{C}^{M_2}$ must satisfy in order to be contained in $\mathbf{C}^{M_1 \,; M_2}$, and prove that there is a bijective correspondence between objects satisfying that property and monad algebras of $\langle M_1 \,; M_2, \eta, \mu \rangle$.

Remark. In the research on semantics of effects, often one uses a distributive law to combine two monads each capable of expressing a different effect into one monad capable of expressing both effects.

Exercise 3. Given $\mathbf{Cat}$-monads $\langle M_1, \eta_1, \mu_1 \rangle$ and $\langle M_2, \eta_2, \mu_2 \rangle$ on a category $\mathbf{C}$, there is a source $\mathbf{C}_{M_1} \xleftarrow{I_1} \mathbf{C} \xrightarrow{I_2} \mathbf{C}_{M_2}$ in $\mathbf{Cat}$. Given a distributive law δ of $\langle M_1, \eta_1, \mu_1 \rangle$ over $\langle M_2, \eta_2, \mu_2 \rangle$ and the resulting monad $\langle M_1 \,; M_2, \eta, \mu \rangle$, show that there is a sink $\mathbf{C}_{M_1} \xrightarrow{I_1'} \mathbf{C}_{M_1 \,; M_2} \xleftarrow{I_2'} \mathbf{C}_{M_2}$ such that $I_1 \,; I_1'$ equals $I_2 \,; I_2'$ and $I : \mathbf{C} \to \mathbf{C}_{M_1 \,; M_2}$ is the diagonal of the corresponding commuting square.

Remark. Note that I_1, I_1', I_2, I_2', and I are all monomorphisms in $\mathbf{Cat}$, i.e. conceptually inclusions of subcategories, if and only if $(\eta_1)_C$, $(\eta_2)_C$, and $(M_2 \eta_1)_C$ are all monic for every object C of $\mathbf{C}$.