Transformations

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Exercise 1. Prove that for any category $\mathbf{C}$ and any object $C : \mathbf{C}$, the category $\mathbf{Sub}(C)$ is thin, meaning there is at most one morphism between any two objects.

Proof. Let $m_1 : S_1 \hookrightarrow C$ and $m_2 : S_2 \hookrightarrow C$ be objects of $\mathbf{Sub}(C)$, and let f_1 and f_2 be morphisms from $\langle S_1, m_1\rangle$ to $\langle S_2, m_2\rangle$. By definition, the means $f_1 \,;\, m_2$ equals m_1 and $f_2 \,;\, m_2$ equals m_1. Thus, $f_1 \,;\, m_2$ equals $f_2 \,;\, m_2$. In order to be object of $\mathbf{Sub}(C)$, m_2 must be a monomorphism. By the definition of monomorphism, the equality $f_1 \,;\, m_2 = f_2 \,;\, m_2$ implies f_1 equals f_2, thereby guaranteeing thinness. □

Exercise 2. Prove that **Prost** is a reflective subcategory of $\mathbf{Rel}(2)$ (the category whose objects are sets with a binary relation and whose morphisms are relation-preserving functions).

Proof. Given a set X with a binary relation $R : X \times X \to \texttt{Prop}$, define $\leq_R$ to be the reflexive-transitive closure of R. The identity function X is a relation-preserving function from $\langle X, R\rangle$ to $\langle X, \leq_R\rangle$ by the definition of closure. Suppose f is a relation-preserving function from $\langle X, R\rangle$ to $\langle Y, \sqsubseteq\rangle$, and $\sqsubseteq$ is a reflexive, transitive relation. Then $f(x) \sqsubseteq f(x)$ due to reflexivity, and given a chain $x_1 \; R \ldots R \; x_n$ we know $f(x_1) \sqsubseteq \cdots \sqsubseteq f(x_n)$ and so $f(x_1) \sqsubseteq f(x_n)$ by transitivity. Therefore, f is also a relation-preserving function from $\langle X, \leq_R\rangle$ to $\langle Y, \sqsubseteq\rangle$ by the definition of reflexive-transitive closure. □

Exercise 3. Suppose a subcategory $\mathbf{S} \overset{I}{\hookrightarrow} \mathbf{C}$ has a mapping from each object $C : \mathbf{C}$ to a reflection arrow $C \xrightarrow{r_C} I(R(C))$. Prove that there is a unique way to extend the function R to a functor from $\mathbf{C}$ to $\mathbf{S}$ such that the reflection arrows form a natural transformation $r : \mathbf{C} \Rightarrow R \,;\, I$.

Proof. Given a $\mathbf{C}$-morphism $f : C_1 \to C_2$, define $R(f)$ to be the unique morphism $(f \,;\, r_{C_2})^{\leftarrow}$ with the property that $r_{C_1} \,;\, I((f \,;\, r_{C_2})^{\leftarrow}) = f \,;\, r_{C_2}$ guaranteed to exist because r_{C_1} is a reflection arrow. By construction, this makes r a natural transformation from $\mathbf{C}$ to $R \,;\, I$. Simililary, uniqueness of $(f \,;\, r_{C_2})^{\leftarrow}$ guarantees uniqueness of R. All that is left to prove is that R is a functor. By definition, $R(f \,;\, g)$ is the unique morphism with the property that $r_{C_1} \,;\, I(R(f \,;\, g))$ equals $(f \,;\, g) \,;\, r_{C_3}$. The chain of equalities $r_{C_1} \,;\, I(R(f) \,;\, R(g)) = r_{C_1} \,;\, I(R(f)) \,;\, I(R(g)) = f \,;\, r_{C_2} \,;\, I(R(g)) = f \,;\, g \,;\, r_{C_3}$ shows that $R(f) \,;\, R(g)$ also enjoys this property and so must equal $R(f \,;\, g)$. Similarly, the chain of equalities $r_C \,;\, I(id_{R(C)}) = r_C \,;\, id_{I(R(C))} = r_C = id_C \,;\, r_C$ implies that $id_{R(C)}$ equals $R(id_C)$. □

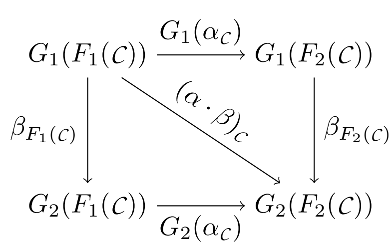
Exercise 4. Prove that the category **Cat** can be enriched in the multicategory **CAT**.

Proof. We present the biased enrichment:

Objects: The class of small categories

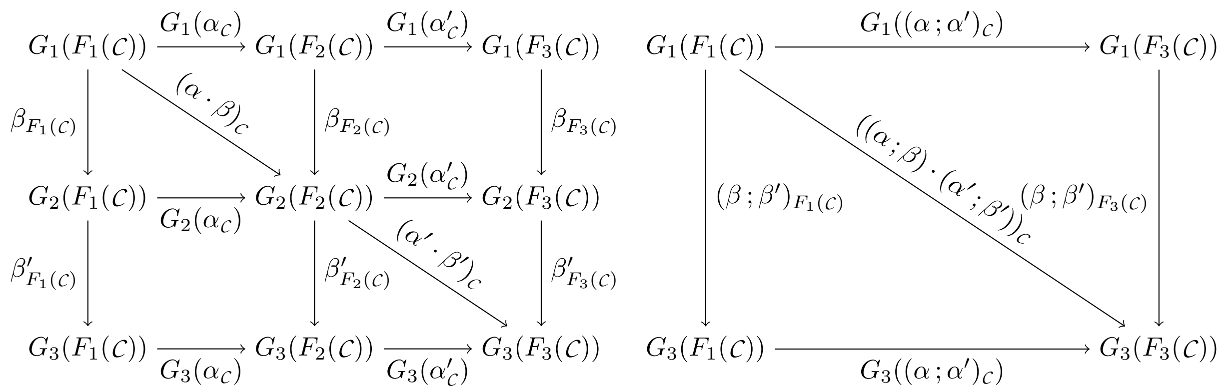
Morphisms: Given small categories $\mathbf{C}$ and $\mathbf{D}$, the corresponding object of morphisms is the category of functors and natural transformations $\mathbf{C} \twoheadrightarrow \mathbf{D}$

Compositions c: Define composition to be the binary functor from $[\mathbf{C} \twoheadrightarrow \mathbf{D}, \mathbf{D} \twoheadrightarrow \mathbf{E}]$ to $\mathbf{C} \twoheadrightarrow \mathbf{E}$ that maps $\langle F : \mathbf{C} \to \mathbf{D}, G : \mathbf{D} \to \mathbf{E}\rangle$ to $F \,;\, G : \mathbf{C} \to \mathbf{E}$ and $\langle \alpha : F_1 \Rightarrow F_2, \beta : G_1 \Rightarrow G_2\rangle$ to $\alpha \cdot \beta : F_1 \,;\, G_1 \Rightarrow F_2 \,;\, G_2$ where $(\alpha \cdot \beta)_C$ is any path in the diagram below, which commutes due to naturality of β:



$\alpha \cdot \beta$ is a natural transformation since $G_1(F_1(\mathfrak{c}))\,;(\alpha \cdot \beta)_{\mathcal{C}_2} = G_1(F_1(\mathfrak{c}))\,;\beta_{F_1(\mathcal{C}_2)}\,;G_2(\alpha_{\mathcal{C}2}) = \beta_{F_1(\mathcal{C}_1)}\,;G_2(F_1(\mathfrak{c}))\,;G_2(\alpha_{\mathcal{C}2}) = \beta_{F_1(\mathcal{C}_1)}\,;G_2(F_1(\mathfrak{c})\,;\alpha_{\mathcal{C}2}) = \beta_{F_1(\mathcal{C}_1)}\,;G_2(\alpha_{\mathcal{C}_1}\,;F_2(\mathfrak{c})) = \beta_{F_1(\mathcal{C}_1)}\,;G_2(\alpha_{\mathcal{C}_1})\,;G_2(F_2(\mathfrak{c})) = (\alpha \cdot \beta)_{\mathcal{C}_1}\,;G_2(F_2(\mathfrak{c}))$ holds for any $\mathfrak{c} : \mathcal{C}_1 \to \mathcal{C}_2$.

To be a functor this process needs to distribute over composition of natural transformations in $\mathbf{C} \multimap \mathbf{D}$ (and preserve identities, which I show later). So, we need to show $(\alpha\,;\alpha') \cdot (\beta\,;\beta')$ equals $(\alpha \cdot \beta)\,;(\alpha' \cdot \beta')$. Consider the following two diagrams:



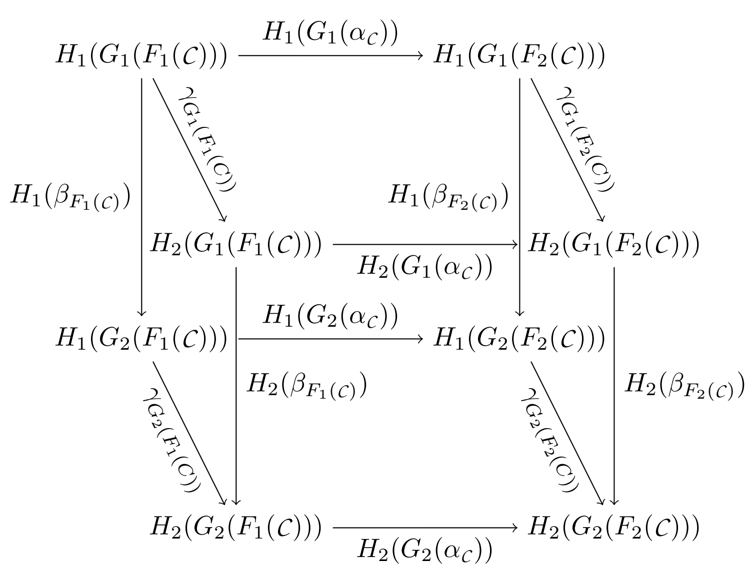
Both diagrams commute due to naturality and the definition of $\cdot$. Notice that the left wall of both diagrams are equal due to the definition of ; on natural transformations and the distributivity of G_1. Similarly for the other three walls. Thus the two diagonals must be equal. Since the left diagrams's diagonal is $((\alpha \cdot \beta)\,;(\alpha' \cdot \beta'))_{\mathcal{C}}$ by definition of ;, this proves $(\alpha\,;\alpha') \cdot (\beta\,;\beta')$ equals $(\alpha \cdot \beta)\,;(\alpha' \cdot \beta')$.

Lastly, this process preserves identities:

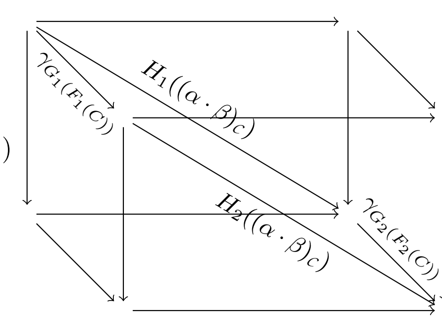
$$(\mathit{id}_F \cdot \mathit{id}_G)_{\mathcal{C}} = G(\mathit{id}_{F(\mathcal{C})})\,;\mathit{id}_{G(F(\mathcal{C}))} = \mathit{id}_{G(F(\mathcal{C}))}\,;\mathit{id}_{G(F(\mathcal{C}))} = \mathit{id}_{G(F(\mathcal{C}))} = (\mathit{id}_{F\,;G})_{\mathcal{C}}$$

Associativity $\mathfrak{a}$: We need to show that $(F\,;G)\,;H$ equals $F\,;(G\,;H)$, which is already known since **Cat** is a category and so composition is associative, and that $(\alpha \cdot \beta) \cdot \gamma$ equals $\alpha \cdot (\beta \cdot \gamma)$. Consider the following cubes:

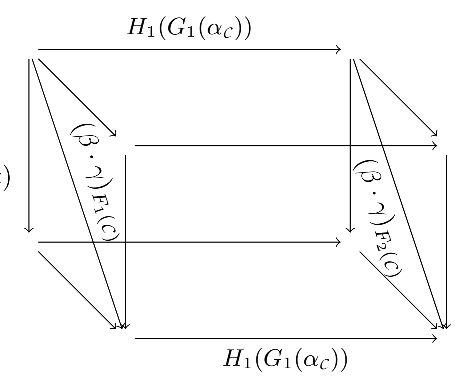
(𝔞)



(𝔟)



(𝔠)



Cube (𝔞) commutes due to naturality and functoriality. It indicates the missing labels for cubes (𝔟) and (𝔠), which also commute by the definition of $\cdot$. $((\alpha \cdot \beta) \cdot \gamma)_{\mathcal{C}}$ is defined to be the diagonal of cube (𝔟), and

$(\alpha \cdot (\beta \cdot \gamma))_C$ is defined to be the diagonal of cube ($\mathfrak{c}$). Both of those are the diagonals of cube ($\mathfrak{a}$) and so must be equal, proving associativity.

Identities i: For each small category $\mathbf{C}$, we need to select an object of $\mathbf{C} \multimap \mathbf{C}$. We select the identity functor.

Identity i: We need to show that $Id_{\mathbf{C}} \,;F = F = F\,; Id_{\mathbf{D}}$, which is true since **Cat** is a category and we are using its identities, and that $Id_{\mathbf{C}} \cdot \alpha = \alpha = \alpha \cdot Id_{\mathbf{D}}$:

$$(id_{Id_{\mathbf{C}}} \cdot \alpha)_C = F_1(id_C)\,; \alpha_{Id_{\mathbf{C}}(C)} = id_{F_1(C)}\,; \alpha_C = \alpha_C = \alpha_C\,; id_{F_2(C)} = Id_{\mathbf{D}}(\alpha_C)\,; id_{Id_{\mathbf{D}}(F_2(C))} = (\alpha \cdot id_{Id_{\mathbf{D}}})_C$$

□