Topoi

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November 30, 2014

Definition. Given an object $\mathcal{C}$ and subobjects $m_1 : \mathcal{S}_1 \hookrightarrow \mathcal{C}$ and $m_2 : \mathcal{S}_2 \hookrightarrow \mathcal{C}$, define $m_1 \subseteq_{\mathcal{C}} m_2$ to be $\exists f : \mathcal{S}_1 \to \mathcal{S}_2.\ m_1 = f \mathbin{;} m_2$.

Theorem. *$\subseteq_{\mathcal{C}}$ is a preorder on the subobjects of $\mathcal{C}$.*

Definition. Given an object $\mathcal{C}$ of a topos, define $\mathbf{true}_{\mathcal{C}} : \mathcal{C} \to \Omega$ to be $\langle\rangle_{\mathcal{C}} \mathbin{;} \mathbf{true}$.

Exercise 1. Given an object $\mathcal{C}$ of a topos and subobjects $m_1 : \mathcal{S}_1 \hookrightarrow \mathcal{C}$ and $m_2 : \mathcal{S}_2 \hookrightarrow \mathcal{C}$, prove that $m_1 \subseteq_{\mathcal{C}} m_2$ holds if and only if $m_1 \mathbin{;} \chi_{m_2}$ equals $\mathbf{true}_{\mathcal{S}_1}$.

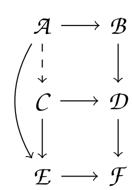
Proof. Suppose $m_1 \subseteq_{\mathcal{C}} m_2$ holds. Let $f : \mathcal{S}_1 \to \mathcal{S}_2$ be a morphism proving this property. Then $m_1 \mathbin{;} \chi_{m_2}$ equals $f \mathbin{;} m_2 \mathbin{;} \chi_{m_2}$, which equals $f \mathbin{;} \langle\rangle_{\mathcal{S}_1} \mathbin{;} \mathbf{true}$, which equals $\langle\rangle_{\mathcal{S}_2} \mathbin{;} \mathbf{true}$, which is the definition of $\mathbf{true}_{\mathcal{S}_1}$.

Suppose $m_1 \mathbin{;} \chi_{m_2}$ equals $\mathbf{true}_{\mathcal{S}_1}$. Then the fact that m_2 is a pullback of χ_{m_2} and $\mathbf{true}$ implies there exists a morphism $f : \mathcal{S}_1 \to \mathcal{S}_2$ such that m equals $f \mathbin{;} m_2$. Thus, f demonstrates that $m_1 \subseteq_{\mathcal{C}} m_2$ holds. □

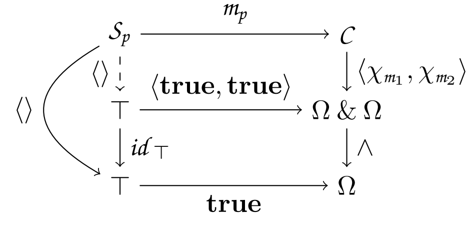
Definition. Given an object $\mathcal{C}$ and a morphism $p : \mathcal{C} \to \Omega$, let $m_p : \mathcal{S}_p \hookrightarrow \mathcal{C}$ be the (unique up to isomorphism) subobject produced by the pullback of $\mathbf{true}$ and p.

Exercise 2. Given an object $\mathcal{C}$ of a topos and subobjects $m_1 : \mathcal{S}_1 \hookrightarrow \mathcal{C}$ and $m_2 : \mathcal{S}_2 \hookrightarrow \mathcal{C}$, let $p : \mathcal{C} \to \Omega$ be defined as $\langle \chi_{m_1}, \chi_{m_2} \rangle \mathbin{;} \wedge$. Prove that m_p is the meet of m_1 and m_2 with respect to the preorder $\mathcal{C}$. Hint: take advantage of the following theorem.

Theorem. *Given any commuting diagram of the following form (minus the dashed line), if the outer $[\mathcal{A}, \mathcal{B}, \mathcal{E}, \mathcal{F}]$ is a pullback square and the lower $[\mathcal{C}, \mathcal{D}, \mathcal{E}, \mathcal{F}]$ is a pullback square, then the upper $[\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}]$ using the uniquely induced dashed line is also a pullback square:*



Proof. Apply the above theorem to the following diagram:



Because the upper square commutes, we have $m_p \mathbin{;} \chi_{m_1} = m_p \mathbin{;} \langle \chi_{m_1}, \chi_{m_2} \rangle \mathbin{;} \pi_1 = \langle\rangle \mathbin{;} \langle \mathbf{true}, \mathbf{true} \rangle \mathbin{;} \pi_1 = \mathbf{true}_{\mathcal{S}_p}$, so by the prior exercise $m_p \subseteq_{\mathcal{C}} m_1$ holds. Similarly, $m_p \subseteq_{\mathcal{C}} m_2$ holds. Thus m_p is a subset of both m_1 and m_2.

Next, suppose there is some subobject $m : \mathcal{S} \hookrightarrow \mathcal{C}$ such that $m \subseteq_{\mathcal{C}} m_1$ and $m \subseteq_{\mathcal{C}} m_2$ hold. Then, by the prior exercise, $m \mathbin{;} \langle \chi_{m_1}, \chi_{m_2} \rangle \mathbin{;} \pi_i = m \mathbin{;} \chi_{m_i} = \langle\rangle \mathbin{;} \mathbf{true} = \langle\rangle \mathbin{;} \langle \mathbf{true}, \mathbf{true} \rangle \mathbin{;} \pi_i$ for both $i \in \{1, 2\}$, which implies $m \mathbin{;} \langle \chi_{m_1}, \chi_{m_2} \rangle$ equals $\langle\rangle \mathbin{;} \langle \mathbf{true}, \mathbf{true} \rangle$ by property of products. Because the upper square is a pullback, this implies there exists a morphism $f : \mathcal{S} \to \mathcal{S}_p$ with the property that m equals $f \mathbin{;} m_p$, proving that $m \subseteq_{\mathcal{C}} m_p$ holds. □