Proofs

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November 21, 2014

Exercise 1. Let $\mathit{trans} : \langle \{a, b, c\}, \{\langle a, b\rangle, \langle b, c\rangle\}\rangle \to \langle \{a, b, c\}, \{\langle a, b\rangle, \langle b, c\rangle, \langle a, c\rangle\}\rangle$ be a morphism of $\mathbf{Rel}(2)$ whose underlying function is the identity. Call its domain $\mathcal{P}$ and its codomain $\mathcal{R}$. Describe, in standard set-theoretic terms, what the pushout of the following is, for any object $\mathcal{X}$ and morphism $f : \mathcal{P} \to \mathcal{X}$:

$$\begin{array}{ccc} \mathcal{P} & \xrightarrow{\mathit{trans}} & \mathcal{R} \\ {\scriptstyle f}\downarrow & & \\ \mathcal{X} & & \end{array}$$

Prove that your description is actually a pushout.

Proof. Let $\mathcal{Y} = \langle X, R_X \cup \{\langle f(a), f(c)\rangle\}\rangle$. Let κ_X be the identity function, which is obviously relation preserving. Let $\kappa_{\mathcal{R}}$ be the underlying function of f, which is relation preserving because f is relation preserving and $f(a)$ is related to $f(c)$ in $\mathcal{Y}$ by definition.

Suppose there is an object $\mathcal{Z}$ with morphisms $g : \mathcal{X} \to \mathcal{Z}$ and $h : \mathcal{R} \to \mathcal{Z}$ such that $f\,;g$ equals $\mathit{trans}\,;h$. Then, for $\kappa_X\,;[g, h]$ to equal g, the underlying function of $[g, h]$ must be the underlying function of g because the underlying function of κ_X is the identity, which guarantees uniqueness of $[g, h]$. This also implies that $\kappa_{\mathcal{R}}\,;[g, h]$ equals h, since the underlying function of $\kappa_{\mathcal{R}}$ is f, and $f\,;g$ equaling $\mathit{trans}\,;h$ implies $f\,;g$ equals h. For existence, we must prove that g is relation-preserving from $\mathcal{Y}$ to $\mathcal{Z}$ and that $\kappa_{\mathcal{R}}\,;g$ equals h. Given two elements related in $\mathcal{Y}$, by the definition of the relation of $\mathcal{Y}$, they must either be related in $\mathcal{X}$ or they must be the pair $\langle f(a), f(c)\rangle$. The former case is preserved because g is relation-preserving from $\mathcal{X}$ to $\mathcal{Z}$. For the latter case, $g(f(a))$ equals $h(a)$ and $g(f(c))$ equals $h(c)$, and a and c are related in $\mathcal{R}$, so h being relation-preserving implies that $g(f(a))$ must be related to $g(f(c))$ in $\mathcal{Z}$. $\square$