Multicategories

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Exercise 1. Prove that there is a multicategory **Mon** whose objects are monoids and whose morphisms are the multilinear homomorphisms with composition and identity are inherited from **Set**. This amounts to defining what nullary multilinear homomorphisms are, and proving that multilinear homomorphisms are closed under composition and identity.

Proof. An n-ary multilinear homomorphism is an n-ary function such that, for each input, fixing all other inputs always produces a monoid homomorphism. Thus, the $n = 1$ case is simply a monoid homomorphism. For the $n > 1$ case, this means fixing any one input to any value produces an $(n-1)$-ary multilinear homomorphism. Since the nullary case has no inputs, this means that all nullary functions are nullary multilinear homomorphisms. Composing with a unary monoid homomorphism produces a nullary function, which is a nullary multilinear homomorphism. Composing with a multi-input multilinear homomorphism effectively fixes an input of the function, which by the definition of a multilinear homomorphism still results in a mulitlinear homomoprhism.

As for composing non-nullary multilinear homomorphisms, I show this produces a multilinear homomorphism for arbitrary $f : [\mathcal{M}_1, \mathcal{M}_2] \to \mathcal{N}_1$ and $g : [\mathcal{N}_1, \mathcal{N}_2] \to \mathcal{O}$ and the remaining non-nullary cases follow from similar arguments. Suppose we have $m_1, m_1' : M_1$, $m_2, m_2' : M_2$, and $n_2, n_2' : N_2$. We need to prove the following:

$g(f(m_1, m_2), e) = e$**:** Follows immediately from multilinearity of g

$g(f(m_1, m_2), n_2 * n_2') = g(f(m_1, m_2), n_2) * g(f(m_1, m_2), n_2')$**:** Follows immediately from multilinearity of g

$g(f(m_1, e), n_2) = e$**:** The left is equal to $g(e, n_2)$ by multilinearity of f, which equals the right by multilinearity of g

$g(f(m_1, m_2 * m_2'), n_2) = g(f(m_1, m_2), n_2) * g(f(m_1, m_2'), n_2')$**:** The left is equal to $g(f(m_1, m_2) * f(m_1, m_2'), n_2)$ by multilinearity of f, which equals the right by multilinearity of g

$g(f(e, m_2), n_2) = e$**:** The left is equal to $g(e, n_2)$ by multilinearity of f, which equals the right by multilinearity of g

$g(f(m_1 * m_1', m_2), n_2) = g(f(m_1, m_2), n_2) * g(f(m_1', m_2), n_2')$**:** The left is equal to $g(f(m_1, m_2) * f(m_1', m_2), n_2)$ by multilinearity of f, which equals the right by multilinearity of g

The identity function is clearly a monoid homomorphism: $\mathit{id}(e) = e$ and $\mathit{id}(m_1 * m_2) = m_1 * m_2 = \mathit{id}(m_1) * \mathit{id}(m_2)$. ☐

Exercise 2. Prove that there is a bijection between the set of categories and the set of pairs $\langle O, M \rangle$ where O is an element of $\mathtt{Type}_1$ and M is a functor of multicategories from $\mathbf{Path}(O)$ to **Set**.

Proof. Suppose we have a category $\mathbf{C}$. Let O be the set $O_\mathbf{C}$. Then objects of $\mathbf{Pair}(O_\mathbf{C})$ are pairs of objects of $\mathbf{C}$. Let the functor M map each pair $\langle \mathcal{A}, \mathcal{B} \rangle$ to $M_\mathbf{C}(A, B)$, the set of $\mathbf{C}$-morphisms from $\mathcal{A}$ to $\mathcal{B}$. Now, suppose $\langle \mathcal{A}, \mathcal{B} \rangle$ is the codomain of a morphism of $\mathbf{Pair}(O_\mathbf{C})$. Then M needs to map this morphism to a function from morphism paths in $\mathbf{C}$ from $\mathcal{A}$ to $\mathcal{B}$ to morphisms in $\mathbf{C}$ from $\mathcal{A}$ to $\mathcal{B}$. Thus, M maps each morphism in $\mathbf{Path}(O_\mathbf{C})$ to the composition operation on the appropriate paths. Distributivity of M is given by associativity of unbiased composition in $\mathbf{C}$, and identity preservation of M is given by identity of unbiased composition in $\mathbf{C}$.

Suppose we have a set O and a functor of multicategories M from $\mathbf{Path}(O)$ to **Set**. Let $O_\mathbf{C}$ be the set O. Let $M_\mathbf{C}(A, B)$ be $M(\langle A, B \rangle)$. For composition, each path has the form $A_1 \to \ldots \to A_n$, so compose a path by applying the function that M maps the unique morphism from $[\langle A_1, A_2 \rangle, \ldots, \langle A_{n-1}, A_n \rangle]$ to $\langle A_1, A_n \rangle$ to. Associativity of unbiased composition is given by distributivity of M and thinness of $\mathbf{Path}(O)$. For example, suppose $c_{A,B,C}$ is the unique morphism in $\mathbf{Path}(O)$ from $[\langle A, B \rangle, \langle B, C \rangle]$ to $\langle A, C \rangle$. Then $c_{A,B,C}$ composed appropriately with $c_{A,C,D}$ equals the unique morphism from $[\langle A, B \rangle, \langle B, C \rangle, \langle C, D \rangle]$ to $\langle A, D \rangle$, and so does $c_{B,C,D}$ composed appropriately with $c_{A,B,D}$. Because of this, distributivity of the functor M implies binary composition is associative. Lastly, unbiased identity is given by the fact that M preserves identities.

These two processes are clearly inverses of each other. ☐

Exercise 3. Give an example of an internal monoid of **Mon** whose underlying set is $\mathbb{N}$.

Proof. Multiplication of natural numbers is a multilinear homomorphism from $[\mathbb{N}_{+,0}, \mathbb{N}_{+,0}]$ to $\mathbb{N}_{+,0}$ that is associative and has $1 : [\,] \to \mathbb{N}_{+,0}$ as its identity. All proofs are basic arithmetic. □

Exercise 4. Define a multicategory **M** with the property that, for any multicategory **C**, there is a bijection between the set of functors from **M** to **C** and the set of internal monoids of **C**.

Proof. Define **M** to be the multicategory $\mathbf{Path}(\mathbb{1})$ with exactly one object and exactly one morphism for each arity. A functor from **M** to **C** picks out a single object $\mathcal{C}$ of **C** and for each arity picks out a $\mathcal{C}$-endomorphism of that arity. Due to the thinness of **M**, functoriality implies that this collection of morphisms satisfies the associativity and identity requirements of unbiased internal monoids, which are in 1-to-1 correspondence with biased internal monoids (via the same proof as for non-internal monoids). For example, if m_n is the unique n-ary morphism in **M**, and o is the binary morphism of **M** that the functor maps m_2, then distributivity implies o is associative since the two ways to compose m_2 with itself to get a ternary morphism both equal the same morphism of **M**, namely m_3, and so must both equal whatever the functor maps m_3 to. □

Exercise 5. Prove that the category **CommMon** can be enriched in the multicategory **CommMon**. That is, show that there is a functor of multicategories from $\mathbf{Path}(O_{\mathbf{CommMon}})$ to **CommMon** that when composed with the underlying functor of multicategories from **CommMon** to **Set** produces the functor of multicategories from $\mathbf{Path}(O_{\mathbf{CommMon}})$ to **Set** defining the category **CommMon**.

Proof. This amounts to imposing a commutative monoidal structure on the set of monoid homomorphisms between any two commutative monoids and proving that composition and identity are multilinear; everything else follows from the fact that **CommMon** is a category. So, given monoid homomorphisms f and g, define $f * g$ to be $\lambda x.\ f(x) * g(x)$. $f * g$ is a monoid homomorphism because $(f * g)(e) = f(e) * g(e) = e * e = e$ and $(f * g)(x * y) = f(x * y) * g(x * y) = f(x) * f(y) * g(x) * g(y) = f(x) * g(x) * f(y) * g(y) = (f * g)(x) * (f * g)(y)$. This operation is commutative and associative because $*$ in the codomain is commutative and associative. Also, define e to be $\lambda x.\ e$, which is trivially a monoid homomorphism. It is the identity of $*$ because e is the identity of $*$ in the codomain.

Next, we need to show that composition is multilinear. $e\,; f_2 = e$ because e always returns the identity element and f_2, being a monoid homomorphism, maps that to the identity. $(f_1 * g_1)\,; f_2 = \lambda x.\ f_2((f_1 * g_1)(x)) = \lambda x.\ f_2(f_1(x) * g_1(x)) = \lambda x.\ f_2(f_1(x)) * f_2(g_1(x)) = (f_1\,; f_2) * (g_1\,; f_2)$. $f_1\,; e = e$ by definition of e. $f_1\,;(f_2 * g_2) = \lambda x.\ (f_2 * g_2)(f_1(x)) = \lambda x.\ f_2(f_1(x)) * g_2(f_1(x)) = (f_1\,; f_2) * (f_1\,; g_2)$.

Lastly, the identity is multilinear because any nullary function is multilinear. □