Monoids

Ross Tate

September 5, 2014

Exercise 1. Given monoids $\mathcal{A}$ and $\mathcal{B}$, give a monoidal structure $\mathcal{A} \,\&\, \mathcal{B}$ to the set $A \times B$ such that the projection functions π_A and π_B are monoid homomorphisms from $\mathcal{A} \,\&\, \mathcal{B}$ to $\mathcal{A}$ and $\mathcal{B}$ respectively.

Proof. Define $\langle a_1, b_1\rangle * \langle a_2, b_2\rangle$ to be $\langle a_1 * a_2, b_1 * b_2\rangle$. This is associative because $*$ is associative for both $\mathcal{A}$ and $\mathcal{B}$. Define $e_{\mathcal{A}\&\mathcal{B}}$ to be $\langle e_{\mathcal{A}}, e_{\mathcal{B}}\rangle$. This is an identity because $e_{\mathcal{A}}$ and $e_{\mathcal{B}}$ are identities for $\mathcal{A}$ and $\mathcal{B}$ respectively. π_A is a monoid homomorphism since $\pi_A(\langle a_1 * a_2, b_1 * b_2\rangle) = a_1 * a_2$, preserving multipliation, and $\pi_A(\langle e_{\mathcal{A}}, e_{\mathcal{B}}\rangle) = e_{\mathcal{A}}$, preserving identity. Similarly for π_B. □

Exercise 2. Determine the monoid "$\top$" with the property that for every monoid $\mathcal{A}$ there is exactly one monoid homomorphism from $\mathcal{A}$ to $\top$.

Proof. The underlying set is $\mathbb{1}$, and multiplication and identity are the only functions with their respective signatures. Given two monoid homomorphisms from some monoid $\mathcal{A}$ to $\top$, they must both map everything to the unique inhabitant of $\mathbb{1}$, making them equal. □

Exercise 3. Determine the monoid "0" with the property that for every monoid $\mathcal{A}$ there is exactly one monoid homomorphism from 0 to $\mathcal{A}$.

Proof. The underlying set is $\mathbb{1}$, and multiplication and identity are the only functions with their respective signatures. Given two monoid homomorphisms from 0 to some monoid $\mathcal{A}$, their only input is the identity of $\top$ and so being monoid homomorphisms they must both map this only input to $e_{\mathcal{A}}$, making them equal. □

Definition. Given monoids $\mathcal{A}$ and $\mathcal{B}$, define the equivalence relation $\approx$ on $\mathbb{L}(A \times B)$ to be the least equivalence relation such that:

1. $\forall \vec{m}_1, \vec{m}'_1, \vec{m}_2, \vec{m}'_2 : \mathbb{L}(A \times B).\ \vec{m}_1 \approx \vec{m}'_1 \wedge \vec{m}_2 \approx \vec{m}'_2 \implies \vec{m}_1 +\!\!+\, \vec{m}_2 \approx \vec{m}'_1 +\!\!+\, \vec{m}'_2$
2. $\forall b : B.\ [\langle e_{\mathcal{A}}, b\rangle] \approx [\,]$
3. $\forall a_1, a_2 : A, b : B.\ [\langle a_1, b\rangle, \langle a_2, b\rangle] \approx [\langle a_1 * a_2, b\rangle]$
4. $\forall a : A.\ [\langle a, e_{\mathcal{B}}\rangle] \approx [\,]$
5. $\forall a : A, b_1, b_2 : B.\ [\langle a, b_1\rangle, \langle a, b_2\rangle] \approx [\langle a, b_1 * b_2\rangle]$

We use requirement 1 to impose a monoidal structure $\mathcal{A} \otimes \mathcal{B}$ on the quotient set $\frac{\mathbb{L}(A\times B)}{\approx}$:

Operator $\frac{+\!+}{\approx} = \lambda q_1, q_2.\ \texttt{select}\ \vec{m}_1\ \texttt{from}\ q_1\ \texttt{in}\ (\texttt{select}\ \vec{m}_2\ \texttt{from}\ q_2\ \texttt{in}\ \frac{\vec{m}_1 +\!+ \vec{m}_2}{\approx}\ \texttt{using .})\ \texttt{using .}$

Associativity Follows from associativity of $+\!+$ and the fact that quotienting only makes things more equal

Identity Element $= \frac{[\,]}{\approx}$

Identity Follows from identity of $[\,]$ and the fact that quotienting only makes things more equal

Exercise 4. Show that, for any monoid $\mathcal{C}$, there is a bijection between the set of multilinear homomorphisms from $\mathcal{A}$ and $\mathcal{B}$ to $\mathcal{C}$ and the set of monoid homomorphisms from $\mathcal{A} \otimes \mathcal{B}$ to $\mathcal{C}$.

Proof. Given a function $f : A \times B \to C$ that is a multilinear homomorphism from $\mathcal{A}$ and $\mathcal{B}$ to $\mathcal{C}$, define $\hat{f} : \mathbb{L}(A \times B) \to C$ to be $\lambda \vec{m}.\ \Pi \texttt{map}_f \vec{m}$ where $\texttt{map}_f$ is the function that takes a list and produces a new list by applying f to each element. $\hat{f}$ is a monoid homomorphism:

- $\hat{f}(\vec{m}_1 \mathbin{++} \vec{m}_2) = \Pi\mathtt{map}_f(\vec{m}_1 \mathbin{++} \vec{m}_2) = \Pi(\mathtt{map}_f\vec{m}_1 \mathbin{++} \mathtt{map}_f\vec{m}_2) = (\Pi\mathtt{map}_f\vec{m}_1) * (\Pi\mathtt{map}_f\vec{m}_2) = \hat{f}(\vec{m}_1) * \hat{f}(\vec{m}_2)$
- $\hat{f}([\,]) = \Pi\mathtt{map}_f[\,] = \Pi[\,] = e_{\mathcal{C}}$

$\hat{f}$ has the property that it maps related lists to equal elements (skipping the additional rules for equivalence relations below):

1. Given $\vec{m}_1, \vec{m}_1', \vec{m}_2, \vec{m}_2' : \mathbb{L}(A \times B)$ such that $\vec{m}_1 \approx \vec{m}_1'$ and $\vec{m}_2 \approx \vec{m}_2'$ hold, by induction on the proof of $\approx$ we can assume $\hat{f}(\vec{m}_1) = \hat{f}(\vec{m}_1')$ and $\hat{f}(\vec{m}_2) = \hat{f}(\vec{m}_2')$. $\hat{f}(m_1 \mathbin{++} m_2) = \hat{f}(\vec{m}_1) * \hat{f}(\vec{m}_2) = \hat{f}(\vec{m}_1') * \hat{f}(\vec{m}_2') = \hat{f}(m_1' \mathbin{++} m_2')$
2. Given $b : B$, $\hat{f}([\langle e_{\mathcal{A}}, b\rangle]) = \Pi\mathtt{map}_f[\langle e_{\mathcal{A}}, b\rangle] = \Pi[f(e_{\mathcal{A}}, b)] = f(e_{\mathcal{A}}, b) = e_{\mathcal{C}} = \Pi[\,] = \Pi\mathtt{map}_f[\,] = \hat{f}([\,])$
3. Given $a_1, a_2 : A$ and $b : B$, $\hat{f}([\langle a_1, b\rangle, \langle a_2, b\rangle]) = \Pi\mathtt{map}_f[\langle a_1, b\rangle, \langle a_2, b\rangle] = \Pi[f(a_1, b), f(a_2, b)] = f(a_1, b) * f(a_2, b) = f(a_1 * a_2, b) = \Pi[f(a_1 * a_2, b)] = \Pi\mathtt{map}_f[\langle a_1 * a_2, b\rangle] = [\langle a_1 * a_2, b\rangle]$
4. Given $a : A$, $\hat{f}([\langle a, e_{\mathcal{B}}\rangle]) = \Pi\mathtt{map}_f[\langle a, e_{\mathcal{B}}\rangle] = \Pi[f(a, e_{\mathcal{B}})] = f(a, e_{\mathcal{B}}) = e_{\mathcal{C}} = \Pi[\,] = \Pi\mathtt{map}_f[\,] = \hat{f}([\,])$
5. Given $a : A$ and $b_1, b_2 : B$, $\hat{f}([\langle a, b_1\rangle, \langle a, b_2\rangle]) = \Pi\mathtt{map}_f[\langle a, b_1\rangle, \langle a, b_2\rangle] = \Pi[f(a, b_1), f(a, b_2)] = f(a, b_1) * f(a, b_2) = f(a, b_1 * b_2) = \Pi[f(a, b_1 * b_2)] = \Pi\mathtt{map}_f[\langle a, b_1 * b_2\rangle] = [\langle a, b_1 * b_2\rangle]$

Consequently, we can define $\tilde{f} : \frac{\mathbb{L}(A\times B)}{\approx} \to C$ to be $\lambda q.\ \mathtt{select}\ \vec{m}\ \mathtt{from}\ q\ \mathtt{in}\ \Pi\mathtt{map}_f\vec{m}\ \mathtt{using}$ (proof above). This is a monoid homomorphism because $\hat{f}$ is a monoid homomorphism.

In the other direction, given a function $g : \frac{\mathbb{L}(A\times B)}{\approx} \to C$ that is a monoid homomorphism from $\mathcal{A} \otimes \mathcal{B}$ to $\mathcal{C}$, define $\bar{g} : A \times B \to C$ to be $\lambda\langle a, b\rangle.\ g(\frac{[\langle a,b\rangle]}{\approx})$. $\bar{g}$ is a multilinear monoid homomorphism from $\mathcal{A}$ and $\mathcal{B}$ to $\mathcal{C}$ since related lists are in equal equivalence classes and g is a monoid homomorphism:

- Given $b : B$, $\bar{g}(e_{\mathcal{A}}, b) = g(\frac{[\langle e_{\mathcal{A}},b\rangle]}{\approx}) = g(\frac{[\,]}{\approx}) = e_{\mathcal{C}}$
- Given $a_1, a_2 : A$ and $b : B$, $\bar{g}(a_1 * a_2, b) = g(\frac{[\langle a_1 * a_2,b\rangle]}{\approx}) = g(\frac{[\langle a_1,b\rangle, \langle a_2,b\rangle]}{\approx}) = g(\frac{[\langle a_1,b\rangle]}{\approx} \frac{++}{\approx} \frac{[\langle a_2,b\rangle]}{\approx}) = g(\frac{[\langle a_1,b\rangle]}{\approx}) * g(\frac{[\langle a_2,b\rangle]}{\approx}) = \bar{g}(a_1, b) * \bar{g}(a_2, b)$
- Given $a : A$, $\bar{g}(a, e_{\mathcal{B}}) = g(\frac{[\langle a,e_{\mathcal{B}}\rangle]}{\approx}) = g(\frac{[\,]}{\approx}) = e_{\mathcal{C}}$
- Given $a : A$ and $b_1, b_2 : B$, $\bar{g}(a, b_1 * b_2) = g(\frac{[\langle a,b_1 * b_2\rangle]}{\approx}) = g(\frac{[\langle a,b_1\rangle, \langle a,b_2\rangle]}{\approx}) = g(\frac{[\langle a,b_1\rangle]}{\approx} \frac{++}{\approx} \frac{[\langle a,b_2\rangle]}{\approx}) = g(\frac{[\langle a,b_1\rangle]}{\approx}) * g(\frac{[\langle a,b_2\rangle]}{\approx}) = \bar{g}(a, b_1) * \bar{g}(a, b_2)$

Given a function $f : A \times B \to C$ that is a multilinear homomorphism from $\mathcal{A}$ and $\mathcal{B}$ to $\mathcal{C}$, we have the following equality for all $a : A$ and $b : B$:

$$\bar{\tilde{f}}(a, b) = \tilde{f}(\frac{[\langle a, b\rangle]}{\approx}) = \mathtt{select}\ \vec{m}\ \mathtt{from}\ \frac{[\langle a, b\rangle]}{\approx}\ \mathtt{in}\ \Pi\mathtt{map}_f\vec{m}\ \mathtt{using}\ \centerdot = \Pi\mathtt{map}_f[\langle a, b\rangle] = \Pi[f(a, b)] = f(a, b)$$

In the other direction, given a function $g : \frac{\mathbb{L}(A\times B)}{\approx} \to C$ that is a monoid homomorphism from $\mathcal{A} \otimes \mathcal{B}$ to $\mathcal{C}$, we have the following equality for all $q : \frac{\mathbb{L}(A\times B)}{\approx}$:

$$\begin{aligned}
g(q) &= \mathtt{select}\ \vec{m}\ \mathtt{from}\ q\ \mathtt{in}\ g(\tfrac{\vec{m}}{\approx})\ \mathtt{using}\ \centerdot \\
&= \mathtt{select}\ \Sigma_i[\langle a_i, b_i\rangle]\ \mathtt{from}\ q\ \mathtt{in}\ g(\tfrac{\Sigma_i[\langle a_i,b_i\rangle]}{\approx})\ \mathtt{using}\ \centerdot \\
&= \mathtt{select}\ \Sigma_i[\langle a_i, b_i\rangle]\ \mathtt{from}\ q\ \mathtt{in}\ \Pi_i g(\tfrac{[\langle a_i,b_i\rangle]}{\approx})\ \mathtt{using}\ \centerdot \\
&= \mathtt{select}\ \Sigma_i[\langle a_i, b_i\rangle]\ \mathtt{from}\ q\ \mathtt{in}\ \Pi\Sigma_i g(\tfrac{[\langle a_i,b_i\rangle]}{\approx})\ \mathtt{using}\ \centerdot \\
&= \mathtt{select}\ \Sigma_i[\langle a_i, b_i\rangle]\ \mathtt{from}\ q\ \mathtt{in}\ \Pi\mathtt{map}_{\lambda\langle a,b\rangle.\ g(\frac{[\langle a,b\rangle]}{\approx})}\Sigma_i[\langle a_i, b_i\rangle]\ \mathtt{using}\ \centerdot \\
&= \mathtt{select}\ \vec{m}\ \mathtt{from}\ q\ \mathtt{in}\ \Pi\mathtt{map}_{\lambda\langle a,b\rangle.\ g(\frac{[\langle a,b\rangle]}{\approx})}\vec{m}\ \mathtt{using}\ \centerdot \\
&= \mathtt{select}\ \vec{m}\ \mathtt{from}\ q\ \mathtt{in}\ \Pi\mathtt{map}_{\bar{g}}\vec{m}\ \mathtt{using}\ \centerdot \\
&= \tilde{\bar{g}}(q)
\end{aligned}$$

□