Limits

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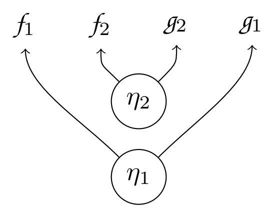
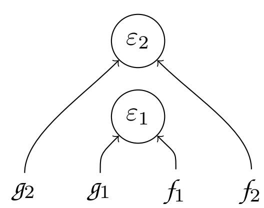
Exercise 1. Prove that for any object $\mathcal{A}$ of any category $\mathbf{C}$, the object $\mathcal{A} \,\&\, \top$ (if it exists) is isomoprhic to $\mathcal{A}$.

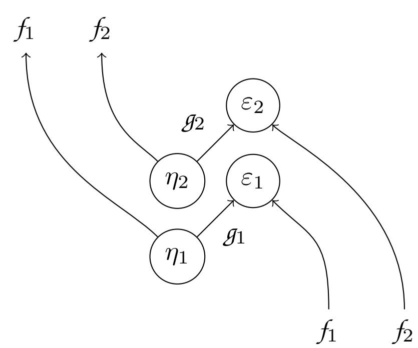
Proof. Let $f : \mathcal{A} \to \mathcal{A} \,\&\, \top$ be defined as $\langle id_{\mathcal{A}}, \langle\rangle\rangle$. Let $g : \mathcal{A} \,\&\, \top \to \mathcal{A}$ be defined as $\pi_{\mathcal{A}}$. $f \,; g = \langle id_{\mathcal{A}}, \langle\rangle\rangle \,; \pi_{\mathcal{A}} = id_{\mathcal{A}}$ by the nature of projections, proving one direction. $g \,; f \,; \pi_{\top} : \mathcal{A} \,\&\, \top \to \top$ equals $id_{\mathcal{A}\&\top} \,; \pi_{\top} : \mathcal{A} \,\&\, \top \to \top$ since all morphisms from the same domain to $\top$ are equal by the nature of terminal objects. Also, $g \,; f \,; \pi_{\mathcal{A}} = \pi_{\mathcal{A}} \,; \langle id_{\mathcal{A}}, \langle\rangle\rangle \,; \pi_{\mathcal{A}} = \pi_{\mathcal{A}} \,; id_{\mathcal{A}} = \pi_{\mathcal{A}} = id_{\mathcal{A}\&\top} \,; \pi_{\mathcal{A}}$. Thus, since all morphisms to a product that behave the same after being followed by both projections must be equal, we have $g \,; f = id_{\mathcal{A}\&\top}$. So, f and g are inverses of each other, making $\mathcal{A}$ isomorphic to $\mathcal{A} \,\&\, \top$. $\square$

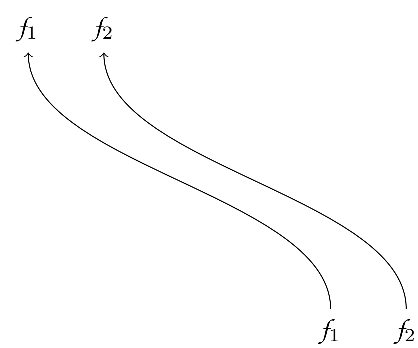
Exercise 2. Prove that, in any 2-category, if morphisms $\mathcal{C}_1 \xrightarrow{f_1} \mathcal{C}_2$ and $\mathcal{C}_2 \xrightarrow{f_2} \mathcal{C}_3$ are both left adjoints, then their composition $f_1 \,; f_2$ is also a left adjoint.

Proof. Let $\langle \mathcal{C}_1, \mathcal{C}_2, f_1, g_1, \eta_1, \varepsilon_1, \mathfrak{f}_1, \mathfrak{g}_1 \rangle$ and $\langle \mathcal{C}_2, \mathcal{C}_3, f_2, g_2, \eta_2, \varepsilon_2, \mathfrak{f}_2, \mathfrak{g}_2 \rangle$ be some adjunctions that f_1 and f_2 are the left adjoints of. Then $\langle \mathcal{C}_1, \mathcal{C}_3, f_1 \,; f_2, g_2 \,; g_1, \eta, \varepsilon, \mathfrak{f}, \mathfrak{g} \rangle$ is an adjuction that $f_1 \,; f_2$ is the left adjoint of, where η, ε, $\mathfrak{f}$, and $\mathfrak{g}$ are defined as follows:

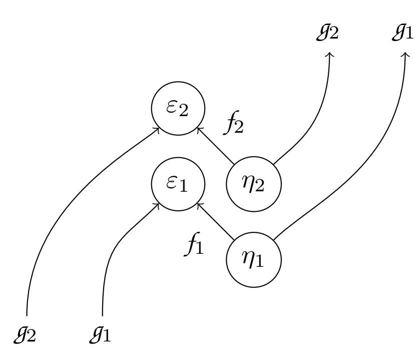
$\eta : \mathcal{C}_1 \Rightarrow f_1 \,; f_2 \,; g_2 \,; g_1 =$ 

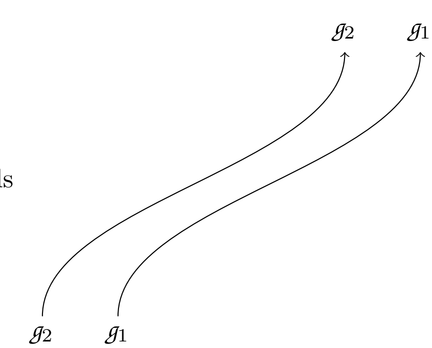
$\varepsilon : g_2 \,; g_1 \,; f_1 \,; f_2 \Rightarrow \mathcal{C}_3 =$ 

$\mathfrak{f}$ is given by 

equals 

due to $\mathfrak{f}_1$ and $\mathfrak{f}_2$.

$\mathfrak{g}$ is given by 

equals 

due to $\mathfrak{g}_1$ and $\mathfrak{g}_2$.

$\square$

Exercise 3. The monoid $\mathcal{A} \mathbin{\&} \mathcal{B}$ is commutative if both $\mathcal{A}$ and $\mathcal{B}$ are commutative, and in that case is (with the appropriate projection homomorphisms) also the product of $\mathcal{A}$ and $\mathcal{B}$ in **CommMon**. Prove that there are morphisms $\kappa_{\mathcal{A}}$ and $\kappa_{\mathcal{B}}$ demonstrating that $\mathcal{A} \mathbin{\&} \mathcal{B}$ is also the coproduct of $\mathcal{A}$ and $\mathcal{B}$ in **CommMon**. That is, prove that **CommMon** has *biproducts*, meaning it has products and coproducts and they coincide on objects.

Proof. Define the underlying function of $\kappa_{\mathcal{A}} : \mathcal{A} \to \mathcal{A} \mathbin{\&} \mathcal{B}$ to be $\lambda a.\ \langle a, e_{\mathcal{B}} \rangle$. This clearly preserves identity and multilplication, making $\kappa_{\mathcal{A}}$ a monoid homomorphism. Similarly, $\kappa_{\mathcal{B}} = \langle \lambda b.\ \langle e_{\mathcal{A}}, b \rangle, \blacksquare, \blacksquare \rangle$.

Given a monoid $\mathcal{C}$ and monoid homomorphisms $f_{\mathcal{A}} : \mathcal{A} \to \mathcal{C}$ and $f_{\mathcal{B}} : \mathcal{B} \to \mathcal{C}$, define the underlying function of $[f] : \mathcal{A} \mathbin{\&} \mathcal{B} \to \mathcal{C}$ to be $\lambda \langle a, b \rangle.\ f_{\mathcal{A}}(a) +_{\mathcal{C}} f_{\mathcal{B}}(b)$. We have to show this is a monoid morphism. It distributes since $[f](\langle a_1, b_1 \rangle +_{\mathcal{A}\&\mathcal{B}} \langle a_2, b_2 \rangle) = [f](\langle a_1 +_{\mathcal{A}} a_2, b_1 +_{\mathcal{B}} b_2 \rangle) = f_{\mathcal{A}}(a_1 +_{\mathcal{A}} a_2) +_{\mathcal{C}} f_{\mathcal{B}}(b_1 +_{\mathcal{B}} b_2) = f_{\mathcal{A}}(a_1) +_{\mathcal{C}} f_{\mathcal{A}}(a_2) +_{\mathcal{C}} f_{\mathcal{B}}(b_1) +_{\mathcal{C}} f_{\mathcal{B}}(b_2) = f_{\mathcal{A}}(a_1) +_{\mathcal{C}} f_{\mathcal{B}}(b_1) +_{\mathcal{C}} f_{\mathcal{A}}(a_2) +_{\mathcal{C}} f_{\mathcal{B}}(b_2) = [f](\langle a_1, b_1 \rangle) +_{\mathcal{C}} [f](\langle a_2, b_2 \rangle)$. It preserves identity since $[f](e_{\mathcal{A}\&\mathcal{B}}) = [f](\langle e_{\mathcal{A}}, e_{\mathcal{B}} \rangle) = f_{\mathcal{A}}(e_{\mathcal{A}}) +_{\mathcal{C}} f_{\mathcal{B}}(e_{\mathcal{B}}) = e_{\mathcal{C}} +_{\mathcal{C}} e_{\mathcal{C}} = e_{\mathcal{C}}$.

Given an $a : A$, $[f](\kappa_{\mathcal{A}}(a)) = [f](\langle a, e_{\mathcal{B}} \rangle) = f_{\mathcal{A}}(a) + f_{\mathcal{B}}(e_{\mathcal{B}}) = f_{\mathcal{A}}(a) + e_{\mathcal{C}} = f_{\mathcal{A}}(a)$, so $\kappa_{\mathcal{A}} ; [f]$ equals $f_{\mathcal{A}}$. Similarly, $\kappa_{\mathcal{B}} ; [f]$ equals $f_{\mathcal{B}}$.

Lastly, suppose $g : \mathcal{A} \mathbin{\&} \mathcal{B} \to \mathcal{C}$ also has the property that $\kappa_{\mathcal{A}} ; g$ equals $f_{\mathcal{A}}$ and $\kappa_{\mathcal{B}} ; g$ equals $f_{\mathcal{B}}$. To be a monoid homomorphism, since for any $a : A$ and $b : B$ the sum $\langle a, e_{\mathcal{B}} \rangle +_{\mathcal{A}\&\mathcal{B}} \langle e_{\mathcal{A}}, b \rangle$ equals $\langle a, b \rangle$, we know that $g(\langle a, b \rangle) = g(\langle a, e_{\mathcal{B}} \rangle +_{\mathcal{A}\&\mathcal{B}} \langle e_{\mathcal{A}}, b \rangle) = g(\kappa_{\mathcal{A}}(a) +_{\mathcal{A}\&\mathcal{B}} \kappa_{\mathcal{B}}(b)) = g(\kappa_{\mathcal{A}}(a)) +_{\mathcal{C}} g(\kappa_{\mathcal{B}}(b)) = f_{\mathcal{A}}(a) +_{\mathcal{C}} f_{\mathcal{B}}(b) = [f](\langle a, b \rangle)$. Thus, any such g must equal $[f]$, proving uniqueness. $\square$