Types

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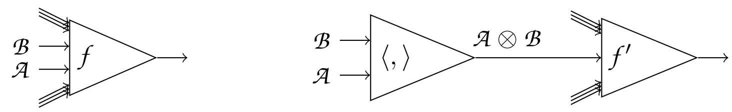
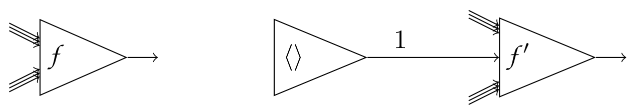
Definition ((Binary) Tensor $\mathcal{A} \otimes \mathcal{B}$ (where $\mathcal{A}$ and $\mathcal{B}$ are objects of a multicategory $\mathbf{M}$))**.** An object of $\mathbf{M}$, denoted $\mathcal{A} \otimes \mathcal{B}$, with a morphism from $[\mathcal{A}, \mathcal{B}]$ to $\mathcal{A} \otimes \mathcal{B}$, denoted (unfortunately) $\langle , \rangle$, such that for any morphism f as in the left diagram below there exists a unique morphism f' as in the right diagram below such that the two diagrams depict equal morphisms.



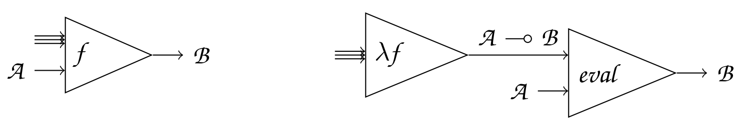
Definition (Nullary Tensor 1 (in a multicategory $\mathbf{M}$))**.** An object of $\mathbf{M}$, denoted 1, with a nullary morphism to 1, denoted (unfortunately) $\langle\rangle$, such that for any morphism f as in the left diagram below there exists a unique morphism f' as in the right diagram below such that the two diagrams depict equal morphisms.



Definition (Representable Multicategory)**.** A multicategory with the property that a nullary tensor exists and a binary tensor exists for all pairs of objects.

Theorem. *There is a 2-functor on* $\mathbf{MULTICAT}$*, called* $\mathbb{L}$*, with a 2-transformation* $[\bullet] : \mathbf{MULTICAT} \Rightarrow \mathbb{L}$*, such that a multicategory* $\mathbf{M}$ *is representable if and only if the 1-cell* $[\bullet]_{\mathbf{M}}$ *is a right adjoint. Given a multicategory* $\mathbf{M}$*, the objects of* $\mathbb{L}(\mathbf{M})$ *are lists of objects of* $\mathbf{M}$*, and the morphisms of* $\mathbb{L}(\mathbf{M})$ *from* $[\vec{\mathcal{C}}_1, \ldots, \vec{\mathcal{C}}_n]$ *to* $\vec{\mathcal{C}}$ *are a list* $\vec{f}$ *of morphisms of* $\mathbf{M}$ *such that the list of outputs of* $\vec{f}$ *equals* $\vec{\mathcal{C}}$ *and flattening the list of lists of inputs of* $\vec{f}$ *equals* $\vec{\mathcal{C}}_1 +\!\!+ \ldots +\!\!+ \vec{\mathcal{C}}_n$*.* $\eta_{\mathbf{M}}$ *is the functor of multicategories mapping an object* $\mathcal{C}$ *of* $\mathbf{M}$ *to the object* $[\mathcal{C}]$ *of* $\mathbb{L}(\mathbf{M})$ *and mapping a morphism* f *to the morphism* $[f]$*, which has the appropriate type because singleton is a unit of flatten.*

Definition (Left Exponential $\mathcal{A} \multimap \mathcal{B}$ (where $\mathcal{A}$ and $\mathcal{B}$ are objects of a multicategory $\mathbf{M}$))**.** An object of $\mathbf{M}$, denoted $\mathcal{A} \multimap \mathcal{B}$, with a morphism from $[\mathcal{A}, \mathcal{A} \multimap \mathcal{B}]$ to $\mathcal{B}$, denoted *eval*, such that for any morphism f as in the left diagram below there exists a unique morphism, denoted λf, as in the right diagram below such that the two diagrams depict equal morphisms.



Definition (Left-Closed Multicategory)**.** A multicategory with the property that left exponentionals exist for all pairs of objects.

Remark. There are analogous definitions for right exponential (denoted $\circ\!\!-$) and right-closed. A multicategory that is both left-closed and right-closed is simply called closed. For symmetric multicategories, left exponentials are always isomorphic to right exponentials, so being left-closed implies being right-closed and vice versa.

Remark. Given a category $\mathbf{C}$ with finite products (including terminal object), one can construct a multicategory $\mathbf{C}_\&$ with the same objects as $\mathbf{C}$ such that a morphism in $\mathbf{C}_\&$ from $[\mathcal{C}_1, \ldots, \mathcal{C}_n]$ to $\mathcal{C}$ is a morphism in $\mathbf{C}$ from $\&_{i:\mathbb{n}} \mathcal{C}_i$ to $\mathcal{C}$. $\mathbf{C}_\&$ will always be representable (and symmetric); the tensor of $\mathcal{A}$ and $\mathcal{B}$ is $\mathcal{A} \,\&\, \mathcal{B}$.

Definition (Cartesian-Closed Category)**.** A category $\mathbf{C}$ with finite products such that $\mathbf{C}_\&$ is closed.

Theorem. *Suppose* $\mathbf{E}$ *is finite. If* $\mathbf{C}$ *has all limits with a finite scheme* $\mathbf{S}$*, then* $\mathbf{E} \rightharpoonup \mathbf{C}$ *has all limits with the scheme* $\mathbf{S}$ *(constructed pointwise). If* $\mathbf{C}$ *has all colimits with a finite scheme* $\mathbf{S}$*, then* $\mathbf{E} \rightharpoonup \mathbf{C}$ *has all colimits with the scheme* $\mathbf{S}$ *(constructed pointwise). If* $\mathbf{C}$ *is finitely complete and cartesian-closed, then* $\mathbf{E} \rightharpoonup \mathbf{C}$ *is also cartesian-closed. Given objects* $\mathcal{A}$ *and* $\mathcal{B}$ *of* $\mathbf{E} \rightharpoonup \mathbf{C}$*, the exponential* $\mathcal{A} \rightharpoonup \mathcal{B}$ *is constructed by the following process. TODO: Twisted arrow category under* $\mathcal{E}$*.*