Topoi

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Definition (Subobject Classifier for a Category $\mathbf{C}$)**.** An object Ω and a morphism $\mathbf{true} : \top \to \Omega$ with the property that, for every monomorphism $m : \mathcal{S} \hookrightarrow \mathcal{C}$ in $\mathbf{C}$, there exists a unique morphism $\chi_m : \mathcal{C} \to \Omega$, called the characteristic morphism of m, with the property that the following is a pullback square:

$$\begin{array}{ccc} \mathcal{S} & \xrightarrow{m} & \mathcal{C} \\ {\scriptstyle \langle\rangle}\downarrow & & \downarrow{\scriptstyle \chi_m} \\ \top & \xrightarrow[\mathbf{true}]{} & \Omega \end{array}$$

Example. $\mathbb{B}$ with $\mathbf{true}$ is the subobject classifier for $\mathbf{Set}$. Given an injection $m : S \rightarrowtail C$, then χ_m is the function $\lambda c.\ \exists s : S.\ m(s) = c$.

Definition (Topos)**.** A finitely complete category with exponentials (with respect to products, denoted $\multimap$) and a subobject classifier.

Example. The category $\mathbf{Set}$ and its full subcategory $\mathbf{Fin}$ of finite sets are both topoi.

Remark. Every morphism from $\top$ is a monomorphism (in any category with a terminal object, not just in topoi).

Theorem. *One can implement $\wedge : \Omega \,\&\, \Omega \to \Omega$ as the characteristic morphism of $\langle \mathbf{true}, \mathbf{true} \rangle : \top \hookrightarrow \Omega \,\&\, \Omega$. One can implement $\Rightarrow : \Omega \,\&\, \Omega \to \Omega$ as the characteristic morphism of the equalizer of π_1 and $\wedge$ from $\Omega \,\&\, \Omega$ to Ω (which works because $\phi \Rightarrow \psi$ holds if and only if $\phi \Leftrightarrow \phi \wedge \psi$ holds).*

Notation. Given a morphism $f : \mathcal{A} \,\&\, \mathcal{B} \to \mathcal{C}$, we denote the corresponding morphism from $\mathcal{B}$ to $\mathcal{A} \multimap \mathcal{C}$ with $\lambda_{\mathcal{A}} f$.

Theorem. *Given an object $\mathcal{C}$, one can implement $\forall_{\mathcal{C}} : (\mathcal{C} \multimap \Omega) \to \Omega$ as the characteristic morphism for $\lambda_{\mathcal{C}}(\pi_2 \,;\mathbf{true}) : \top \hookrightarrow (\mathcal{C} \multimap \Omega)$.*

Theorem. *One can implement $\mathbf{false} : \top \to \Omega$ as the morphism $(\lambda_{\Omega}\pi_1) \,; \forall_{\Omega}$ (which represents the proposition $\forall \phi : \mathtt{Prop}.\ \phi$).*

Theorem. *The pullback of $\mathbf{true} : \top \to \Omega$ and $\mathbf{false} : \top \to \Omega$ is an initial object.*

Theorem. *One can use the above components to implement $\vee : \Omega \,\&\, \Omega \to \Omega$ via the predicate $\forall p : \Omega.\ (\phi \Rightarrow p) \wedge (\psi \Rightarrow p) \Rightarrow p$. Similarly, one can implement $\exists_{\mathcal{C}} : (\mathcal{C} \multimap \Omega) \to \Omega$ via the predicate $\forall p : \Omega.\ (\forall c : \mathcal{C}.\ \phi(c) \Rightarrow p) \Rightarrow p$.*

Theorem. *One can implement $=_{\mathcal{C}} : \mathcal{C} \,\&\, \mathcal{C} \to \Omega$ as the characteristic morphism of $\langle id_{\mathcal{C}}, id_{\mathcal{C}} \rangle : \mathcal{C} \hookrightarrow \mathcal{C} \,\&\, \mathcal{C}$.*

Definition (Natural-Numbers Object of a Category $\mathbf{C}$)**.** An object $\mathcal{N}$ along with morphisms $z : \top \to \mathcal{N}$ and $s : \mathcal{N} \to \mathcal{N}$ with the property that, for every object $\mathcal{C}$ and morphisms $c_z : \top \to \mathcal{C}$ and $c_s : \mathcal{C} \to \mathcal{C}$, there exists a unique morphism $ind(c_z, c_s) : \mathcal{N} \to \mathcal{C}$ such that the following commutes:

$$\begin{array}{cccc} & \overset{z}{\nearrow} \mathcal{N} & \xrightarrow{s} & \mathcal{N} \\ \top & \quad\downarrow{\scriptstyle ind(c_z, c_s)} & & \downarrow{\scriptstyle ind(c_z, c_s)} \\ & \underset{c_z}{\searrow} \mathcal{C} & \xrightarrow[c_s]{} & \mathcal{C} \end{array}$$

Example. $\mathbb{N}$ with 0 and $\lambda n.\ n + 1$ is a natural-numbers object of $\mathbf{Set}$. $\mathbf{Fin}$ has no natural-numbers object.

Theorem. *All topoi are finitely cocomplete.*

Definition (Boolean Topos)**.** A topos with the property that $\top \xrightarrow{\mathbf{true}} \Omega \xleftarrow{\mathbf{false}} \top$ is a coproduct.

Definition (Two-Value Topos)**.** A topos with exactly two morphisms from $\top$ to Ω (necessarily $\mathbf{true}$ and $\mathbf{false}$).

Definition (Well-Pointed)**.** The property that for all $f, g : \mathcal{C}_1 \to \mathcal{C}_2$, $\forall e : \top \to \mathcal{C}_1.\ e \,; f = e \,; g$ implies f equals g.'

Definition (Topos admitting the Axiom of Choice)**.** A topos with the property that all epimorphisms are sections.

Theorem. *Every topos admitting the axiom of choice is Boolean. Every well-pointed topos is two-value. Every well-pointed topos is Boolean (using a classical metatheory).*