Multicategories

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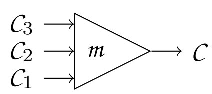
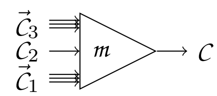
Definition ((Biased) Multicategory)**.** A tuple $\langle O, M, ;, \mathfrak{c}, \mathfrak{a}, \mathit{id}, \mathfrak{i}\rangle$ whose components have the following types:

Objects O**:** $\texttt{Type}_1$

Morphisms M**:** $\mathbb{L}O \times O \to \texttt{Type}$

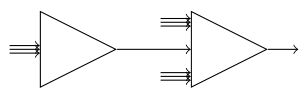


A morphism m from $[\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3]$ to $\mathcal{C}$



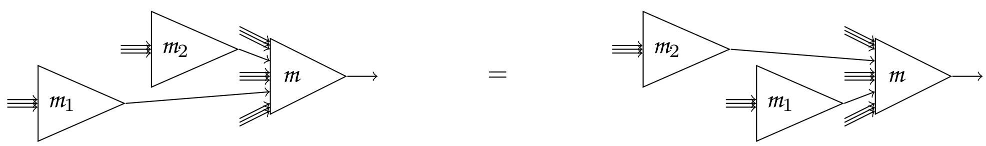
A morphism m from $\vec{\mathcal{C}}_1 +\!\!+ [\mathcal{C}_2] +\!\!+ \vec{\mathcal{C}}_3$ to $\mathcal{C}$

Composition $;$**:** $\forall \mathcal{C}_2, \mathcal{C} : O, \vec{\mathcal{C}}_1, \vec{\mathcal{C}}_2, \vec{\mathcal{C}}_3 : \mathbb{L}O.\ M(\vec{\mathcal{C}}_2, \mathcal{C}_2) \times M(\vec{\mathcal{C}}_1 +\!\!+ [\mathcal{C}_2] +\!\!+ \vec{\mathcal{C}}_3, \mathcal{C}) \to M(\vec{\mathcal{C}}_1 +\!\!+ \vec{\mathcal{C}}_2 +\!\!+ \vec{\mathcal{C}}_3, \mathcal{C})$

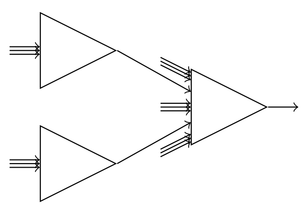


The depiction of the composition of two morphisms

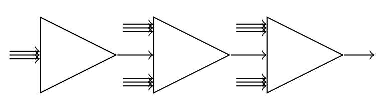
Commutativity $\mathfrak{c}$**:** The following two depictions are always equal:



In other words, the two ways to compose the morphisms in the following diagram together are equal:

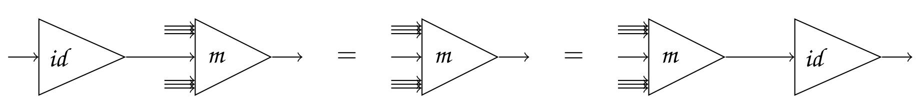


Associativity $\mathfrak{a}$**:** The two ways to compose the morphisms in the following diagram are equal:



Identities id**:** $\forall \mathcal{C} : O.\ M([\mathcal{C}], \mathcal{C})$

Identity $\mathfrak{i}$**:** The following equalities always hold:



Definition (Functor of Multicategories)**.** A mapping of objects and morphisms such that domain, codomain, composition, and identity are preserved.

Definition (**MCAT**)**.** The category whose objects are multicategories and whose morphisms are functors between those multicategories, with composition and identity being the obvious.

Exercise 1. Show that there is a functor from **CAT** to **MCAT** and another from **MCAT** to **CAT** such that their composition is the identity on **CAT**.

Example. Set: Objects are types and morphisms are functions with multiple inputs and one output.

Prost: Objects are preordered sets and morphisms from $[\langle S_1, R_1\rangle, \ldots, \langle S_n, R_n\rangle]$ to $\langle S, R\rangle$ are functions $f : S_1 \times \cdots \times S_n \rightarrow S$ such that $\forall n : \mathbb{N}, (s_i : S_i)_{i:\mathbb{n}}, (s_i : S_i)_{i:\mathbb{n}}.\ (\forall i : \mathbb{n}.\ s_i\ R_i\ s'_i) \implies f(s_1, \ldots, s_n)\ R\ f(s'_1, \ldots, s'_n)$

Cat and CAT: Objects are (small) categories and morphisms are multiple-input functors F, where preservation of composition and identity means that $F(m_1, \ldots, m_n) ; F(m'_1, \ldots, m'_n) = F(m_1 ; m'_1, \ldots, m_n ; m'_n)$ and $id = F(id_1, \ldots, id_n)$.

$\mathcal{M}$ **where** $\mathcal{M}$ **is a monoid:** Objects are elements of M and there exists a unique morphism from $\vec{m}$ to m if and only if $\Pi\vec{m} = m$.

Exercise 2. Show that there are two functors of multicategories, *Obj* and *Mor*, from **Cat** to **Set**, with the first mapping a small category to its set of objects and the second mapping a small category to its set of morphisms.

Exercise 3. Show that one can define a monoidal structure on the objects of a monoidal category if for every list of objects there exists a unique morphism with that list as its domain.

Example. For the following examples we use a common pattern. Each object will have associated with it a left "anchor" and right "anchor". The morphisms will exist from a list of objects to an object only if:

- for each pair of adjacent input objects, the right anchor of the first is equal to the left anchor of the second
- the left anchor of the first input equals the left anchor of the output and the right anchor of the last input equals the right anchor of the output *or* there is no input and the left and right anchors of the output are equal

We present the first such example in more formal detail to help clarify this.

Path$(A : \texttt{Type}_1)$**:** An object is a pair of As and there exists a unique morphism from $[\langle a_1, a'_1\rangle, \ldots, \langle a_n, a'_n\rangle]$ to $\langle a'_0, a_{n+1}\rangle$ if and only if $\forall i : \mathbb{n} \cup \{0\}.\ a'_i = a_{i+1}$.

SplitGraph: An object is a split graph $S \xleftarrow{s} E \xrightarrow{t} T$ where S is the type of (source) nodes, T is the type of (target) nodes, E is the type of edges, and the functions s and t specifying the source and target of each edge. A morphism from $S_1 \xleftarrow{s_1} E_1 \xrightarrow{t_1} (T_1 = S_2) \xleftarrow{s_2} E_2 \xrightarrow{t_2} (T_2 = \cdots = S_n) \xleftarrow{s_n} E_n \xrightarrow{t_n} T_n$ to $S_1 \xleftarrow{s} E \xrightarrow{t} T_n$ is a function mapping each chain of edges $s_1 \xrightarrow{e_1} (t_1 = s_2) \xrightarrow{e_2} (t_2 = \cdots = s_n) \xrightarrow{e_n} t_n$ to an edge $e : E$ with $s(e) = s_1$ and $t(e) = t_n$.

BinRel: An object is a binary relation $R : S \times T \rightarrow \texttt{Prop}$. A morphism from $[R_0 : S_0 \times S_1 \rightarrow \texttt{Prop}, \ldots, R_n : S_n \times S_{n+1} \rightarrow \texttt{Prop}]$ to $R : S_0 \rightarrow S_{n+1}$ both exists and is unique if and only if the following holds:

$$\forall s_0 : S_0, s_{n+1} : S_{n+1}.\ \exists s_1 : S_1, \ldots, s_n : S_n.\ R_0(s_0, s_1) \wedge \cdots \wedge R_n(s_n, s_{n+1}) \implies R(s_0, s_{n+1})$$

Exercise 4. Show that there are functors of multicategories from **SplitGraph** and from **BinRel** to **Path**(`Type`).

Exercise 5. Show that there is a functor of multicategories from **BinRel** to **SplitGraph** and from **SplitGraph** to **BinRel** such that their composition equals the identity on **BinRel**.

Exercise 6. Show that there is a functor of categories from **Cat** to **SplitGraph** (restricted to unary morphisms) that does not extend to a functor of multicategories from **Cat** to **SplitGraph**.