Functors

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Definition ((Biased) (**Set**-enriched) (Covariant) Functor from $\langle O_{\mathbf{C}}, M_{\mathbf{C}}, ;_{,} \blacksquare, \mathit{id}_{\mathbf{C}}, \blacksquare\rangle$ to $\langle O_{\mathbf{D}}, M_{\mathbf{D}}, ;_{\mathbf{D}}, \blacksquare, \mathit{id}_{\mathbf{D}}, \blacksquare\rangle$)**.** A tuple $\langle F_O, F_M, \mathfrak{d}, \mathfrak{i}\rangle$ where the components have the following types:

Function on Objects F_O**:** $O_{\mathbf{C}} \to O_{\mathbf{D}}$

Function on Morphisms F_M**:** $\forall \mathcal{C}_1, \mathcal{C}_2 : O_{\mathbf{C}}.\ M_{\mathbf{C}}(\mathcal{C}_1, \mathcal{C}_2) \to M_{\mathbf{D}}(F_O(\mathcal{C}_1), F_O(\mathcal{C}_2))$ (objects implicit)

Distributivity $\mathfrak{d}$**:** $\forall \mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3 : O_{\mathbf{C}}.\ \forall m_1 : M_{\mathbf{C}}(\mathcal{C}_1, \mathcal{C}_2), m_2 : M_{\mathbf{C}}(\mathcal{C}_2, \mathcal{C}_3).\ F_M(m_1) ;_{\mathbf{D}} F_M(m_2) = F_M(m_1 ;_{\mathbf{C}} m_2)$

Identity $\mathfrak{i}$**:** $\forall \mathcal{C} : O_{\mathbf{C}}.\ \mathit{id}_{\mathbf{D}} = F_M(\mathit{id}_{\mathbf{D}})$

Notation. We represent both F_O and F_M with just F.

Notation. When multiple categories (such as **C** and **D**) are present, we represent all their morphism types (e.g. $M_{\mathbf{C}}$ and $M_{\mathbf{D}}$) with just M or with the infix operator $\to$.

Notation. When multiple categories (such as **C** and **D**) are present, we represent all their composition operators (e.g. $;_{\mathbf{C}}$ and $;_{\mathbf{D}}$) with just $;$.

Notation. When multiple categories (such as **C** and **D**) are present, we represent all their identity operators (e.g. $\mathit{id}_{\mathbf{C}}$ and $\mathit{id}_{\mathbf{D}}$) with just id.

Notation. We use $\mathcal{C} : \mathbf{C}$ to denote $\mathcal{C} : O_{\mathbf{C}}$.

Remark. With these notations, the above definitions can be rephrased as: A functor from **C** to **D** is a tuple $\langle F, F, \mathfrak{d}, \mathfrak{i}\rangle$ where the components have the following types:

Function on Objects F**:** $O_{\mathbf{C}} \to O_{\mathbf{D}}$

Function on Morphisms F**:** $\forall \mathcal{C}_1, \mathcal{C}_2 : \mathbf{C}.\ \mathcal{C}_1 \to \mathcal{C}_2 \to F(\mathcal{C}_1) \to F(\mathcal{C}_2)$ (objects implicit)

Distributivity $\mathfrak{d}$**:** $\forall \mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3 : \mathbf{C}.\ \forall \mathcal{C}_1 \xrightarrow{m_1} \mathcal{C}_2, \mathcal{C}_2 \xrightarrow{m_2} \mathcal{C}_3.\ F(m_1) ; F(m_2) = F(m_1 ; m_2)$

Identity $\mathfrak{i}$**:** $\forall \mathcal{C} : \mathbf{C}.\ \mathit{id} = F(\mathit{id})$

Example. Prost to Set: $\langle \lambda\tau.\ \tau, \lambda\langle\tau_1, \tau_2\rangle.\ \lambda\langle f, \blacksquare\rangle.\ f, \blacksquare, \blacksquare\rangle$

Mat to Set: $\langle \lambda n.\ \mathbb{R}^n, \lambda\langle n_1, n_2\rangle.\ \lambda M.\ \lambda\vec{x}.\ M \cdot \vec{x}, \blacksquare, \blacksquare\rangle$

$\mathbb{\Delta}$ **to Prost:** $\langle \lambda n.\ \langle \mathbb{n}, \leq, \blacksquare, \blacksquare\rangle, \lambda\langle n_1, n_2\rangle.\ \lambda\sigma.\ \langle\sigma, \blacksquare\rangle, \blacksquare, \blacksquare\rangle$

ω **to** $\mathbb{\Delta}$**:** $\langle \lambda n.\ n, \lambda\langle n_1, n_2\rangle.\ \lambda\mathfrak{Ite}.\ \lambda n.\ n$ as $\mathbb{n}_2$ using $\mathfrak{Ite}, \blacksquare, \blacksquare\rangle$

Set to Rel: $\langle \lambda\tau.\ \tau, \lambda\langle\tau_1, \tau_2\rangle.\ \lambda f.\ \lambda\langle t_1, t_2\rangle.\ f(t_1) = t_2, \blacksquare, \blacksquare\rangle$

Rel to Set: $\langle \mathbb{P}, \lambda\langle\tau_1, \tau_2\rangle.\ \lambda\phi.\ \lambda T_1.\ \{t_2 : \tau_2 \mid \exists t_1 : \tau_1.\ t_1 \in T_1 \wedge \phi(t_1, t_2)\}, \blacksquare, \blacksquare\rangle$

$\mathbf{Mon}_b$ **to** $\mathbf{Alg}(2, 0)$**:** $\langle \lambda\langle M, *, \blacksquare, e, \blacksquare\rangle.\ \langle M, *, e\rangle, \lambda\langle\mathcal{M}_1, \mathcal{M}_2\rangle.\ \lambda\langle f, \blacksquare, \blacksquare\rangle.\ \langle f, \blacksquare\rangle, \blacksquare, \blacksquare\rangle$

$\mathbf{Mon}_u$ **to** $\mathbf{Alg}(\langle\mathbb{N}, \lambda n.\ \mathbb{n}\rangle)$**:** $\langle \lambda\langle M, \prod, \blacksquare\rangle.\ \langle M, \lambda n.\ \lambda(m_i)_{i\in\mathbb{n}}.\ \prod[m_1, \ldots, m_n], \blacksquare\rangle, \lambda\langle\mathcal{M}_1, \mathcal{M}_2\rangle.\ \lambda\langle f, \blacksquare\rangle.\ \langle f, \blacksquare\rangle, \blacksquare, \blacksquare\rangle$

Prost to $\mathbf{Rel}(2)$**:** $\langle \lambda\langle\tau, R, \blacksquare\rangle.\ \langle\tau, R\rangle,\ \lambda f.\ f, \blacksquare, \blacksquare\rangle$

Exercise 1. Prove that there are functors from **Set** to **Set** mapping a set S to $\mathbb{L}S$, to $\mathbb{M}S$, to $\mathbb{S}S$, and to $\mathbb{P}S$.

Exercise 2. Prove that the set of functors from **1** to **C** is isomorphic to O_C.

Exercise 3. Prove that the set of functors from **2** to **C** is isomorphic to $\sum_{\mathcal{C}_1, \mathcal{C}_2 : \mathbf{C}} \mathcal{C}_1 \to \mathcal{C}_2$.

Exercise 4. Prove that there is a category **CAT** (of a higher universe) of categories whose objects are categories and whose morphisms are functors, such that there exists a functor from **CAT** to **SET** (i.e. **Set** for a higher universe) mapping categories to their set of objects and functors to their functions on objects.

Definition (**Cat**)**.** The category of "small" categories (categories where O is an element of `Type` rather than $\texttt{Type}_1$) and functors between them.

Definition ((Biased) (**Set**-enriched) Contravariant Functor from $\langle O_{\mathbf{C}}, M_{\mathbf{C}}, ;, \bullet, id_{\mathbf{C}}, \bullet\rangle$ to $\langle O_{\mathbf{D}}, M_{\mathbf{D}}, ;_{\mathbf{D}}, \bullet, id_{\mathbf{D}}, \bullet\rangle$)**.** A tuple $\langle F_O, F_M, \mathfrak{d}, \mathfrak{i}\rangle$ where the components have the following types:

Function on Objects F**:** $O_{\mathbf{C}} \rightarrow O_{\mathbf{D}}$

Function on Morphisms F**:** $\forall C_1, C_2 : \mathbf{C}.\ C_1 \to C_2 \rightarrow F(C_2) \to F(C_1)$ (objects implicit)

Distributivity $\mathfrak{d}$**:** $\forall C_1, C_2, C_3 : \mathbf{C}.\ \forall C_1 \xrightarrow{m_1} C_2, C_2 \xrightarrow{m_2} C_3.\ F(m_2) \,;\, F(m_1) = F(m_1 \,;\, m_2)$

Identity $\mathfrak{i}$**:** $\forall C : \mathbf{C}.\ id = F(id)$

Exercise 5. Prove that there is a contravariant functor from **Set** to **Set** mapping a set S to $\mathbb{P}S$.

Exercise 6. Prove that there is a contravariant functor from **Rel** to **Rel** whose object is component is the identity function.

Exercise 7. Prove that there is a contravariant functor from **Sig** to **CAT** mapping Ω : **Sig** to **Alg**(Ω) : **CAT**.

Exercise 8. Prove that there is a contravariant functor from **Sig** to **CAT** mapping Φ : **Sig** to **Rel**(Φ) : **CAT**.