Aggregation

Ross Tate

August 28, 2014

Definition ((Unbiased) Monoid)**.** A tuple $\langle M, \Pi, \mathfrak{a}, \mathfrak{i} \rangle$ where the components have the following types:

Underlying Set M**:** `Type`

Aggregator Π**:** $\mathbb{L}M \to M$

Associativity $\mathfrak{a}$**:** $\forall n : \mathbb{N},\ \vec{m}_1, \ldots, \vec{m}_n : \mathbb{L}M.\ \Pi\,[\Pi\vec{m}_1, \ldots, \Pi\vec{m}_n] = \Pi(\vec{m}_1 +\!\!+ \ldots +\!\!+ \vec{m}_n)$

Identity $\mathfrak{i}$**:** $\forall m : M.\ m = \Pi\,[m]$

Notation. We use $m_1 * \cdots * m_n$ to denote $\prod\,[m_1, \ldots, m_n]$.

Remark. This definition provides an n-ary operator for every n, which is why we call it *unbiased*. The former definition provided an operator for only arities 0 and 2, which is why we call it *biased*.

Example. $\mathbb{N}_\Sigma = \langle \mathbb{N}, \Sigma, \mathbf{.}, \mathbf{.} \rangle$

$\mathbb{N}_\Pi = \langle \mathbb{N}, \Pi, \mathbf{.}, \mathbf{.} \rangle$

$\mathbb{N}_{\max} = \langle \mathbb{N}, \max, \mathbf{.}, \mathbf{.} \rangle$

$\mathbb{Z}_\Sigma = \langle \mathbb{Z}, \Sigma, \mathbf{.}, \mathbf{.} \rangle$

$\mathbb{Z}_\Pi = \langle \mathbb{Z}, \Pi, \mathbf{.}, \mathbf{.} \rangle$

$\mathbb{R}_\Sigma = \langle \mathbb{R}, \Sigma, \mathbf{.}, \mathbf{.} \rangle$

$\mathbb{R}_\Pi = \langle \mathbb{R}, \Pi, \mathbf{.}, \mathbf{.} \rangle$

$\mathbb{N}^\infty_\Sigma = \langle \mathbb{N}^\infty, \Sigma, \mathbf{.}, \mathbf{.} \rangle$ where $\forall \vec{n} : \mathbb{L}(\mathbb{N}^\infty).\ \infty \in \vec{n} \implies \Sigma\vec{n} = \infty$

$\mathbb{N}^\infty_{\min} = \langle \mathbb{N}^\infty, \min, \mathbf{.}, \mathbf{.} \rangle$ where $\forall \vec{n} : \mathbb{L}(\mathbb{N}^\infty).\ \min \vec{n} = \infty \implies \vec{n} = [\infty, \ldots, \infty]$ (including [ ])

$\mathbb{B}_\wedge = \langle \mathbb{B}, \wedge, \mathbf{.}, \mathbf{.} \rangle$

$\mathbb{B}_\vee = \langle \mathbb{B}, \vee, \mathbf{.}, \mathbf{.} \rangle$

$(\mathbb{L}T)_\Sigma = \langle \mathbb{L}T, \Sigma, \cdot, \cdot \rangle$ where $\Sigma\left[\vec{t}_1, \ldots, \vec{t}_n\right] = \vec{t}_1 +\!\!+ \ldots +\!\!+ \vec{t}_n$ ([ ] when $n = 0$)

$(\mathbb{M}T)_\Sigma = \langle \mathbb{M}T, \Sigma, \cdot, \cdot \rangle$

$(\mathbb{S}T)_\cup = \langle \mathbb{S}T, \cup, \cdot, \cdot \rangle$

$(\mathbb{P}T)_\cup = \langle \mathbb{P}T, \cup, \cdot, \cdot \rangle$

$(\mathbb{P}T)_\cap = \langle \mathbb{P}T, \cap, \cdot, \cdot \rangle$

Exercise 1. Give a bijection between unbiased monoids and biased monoids.

Remark. The mapping of underlying sets and operations is fairly straightforward, but proving the associativity and identity laws is challenging, especially in the biased-to-unbiased direction.

Notation. We call the function from unbiased monoids to biased monoids *Bias*, and the inverse *Unbias*.