Topoi

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Definition. Given an object $\mathcal{C}$ and subobjects $m_1 : \mathcal{S}_1 \hookrightarrow \mathcal{C}$ and $m_2 : \mathcal{S}_2 \hookrightarrow \mathcal{C}$, define $m_1 \subseteq_{\mathcal{C}} m_2$ to be $\exists f : \mathcal{S}_1 \to \mathcal{S}_2.\ m_1 = f \,;\, m_2$.

Theorem. *$\subseteq_{\mathcal{C}}$ is a preorder on the subobjects of $\mathcal{C}$.*

Definition. Given an object $\mathcal{C}$ of a topos, define $\mathbf{true}_{\mathcal{C}} : \mathcal{C} \to \Omega$ to be $\langle\rangle_{\mathcal{C}} \,;\, \mathbf{true}$.

Exercise 1. Given an object $\mathcal{C}$ of a topos and subobjects $m_1 : \mathcal{S}_1 \hookrightarrow \mathcal{C}$ and $m_2 : \mathcal{S}_2 \hookrightarrow \mathcal{C}$, prove that $m_1 \subseteq_{\mathcal{C}} m_2$ holds if and only if $m_1 \,;\, \chi_{m_2}$ equals $\mathbf{true}_{\mathcal{S}_1}$.

Definition. Given an object $\mathcal{C}$ and a morphism $p : \mathcal{C} \to \Omega$, let $m_p : \mathcal{S}_p \hookrightarrow \mathcal{C}$ be the (unique up to isomorphism) subobject produced by the pullback of **true** and p.

Exercise 2. Given an object $\mathcal{C}$ of a topos and subobjects $m_1 : \mathcal{S}_1 \hookrightarrow \mathcal{C}$ and $m_2 : \mathcal{S}_2 \hookrightarrow \mathcal{C}$, let $p : \mathcal{C} \to \Omega$ be defined as $\langle \chi_{m_1}, \chi_{m_2} \rangle \,;\, \wedge$. Prove that m_p is the meet of m_1 and m_2 with respect to the preorder $\mathcal{C}$. Hint: take advantage of the following theorem.

Theorem. *Given any commuting diagram of the following form (minus the dashed line), if the outer $[\mathcal{A}, \mathcal{B}, \mathcal{E}, \mathcal{F}]$ is a pullback square and the lower $[\mathcal{C}, \mathcal{D}, \mathcal{E}, \mathcal{F}]$ is a pullback square, then the upper $[\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}]$ using the uniquely induced dashed line is also a pullback square:*