In order to extend our denotational semantics to higher-order constructs, we will need to develop the theory of complete partial orders (CPOs) and continuous functions on them.

1 Partial Orders

A binary relation $\sqsubseteq$ on a set S is called a *partial order* if it is

- *reflexive*: for all $x \in S$, $x \sqsubseteq x$;
- *transitive*: for all $x, y, z \in S$, if $x \sqsubseteq y$ and $y \sqsubseteq z$, then $x \sqsubseteq z$; and
- *antisymmetric*: for all $x, y \in S$, if $x \sqsubseteq y$ and $y \sqsubseteq x$, then $x = y$.

A partial order $\sqsubseteq$ is a *total order* if for all $x, y \in S$, either $x \sqsubseteq y$ or $y \sqsubseteq x$. A pair of elements $x, y \in S$ are called *comparable* if either $x \sqsubseteq y$ or $y \sqsubseteq x$, *incomparable* otherwise. Thus a total order is one in which all pairs of elements are comparable.

A set with a distinguished partial order defined on it, $(S, \sqsubseteq)$, is called a *partially ordered set* or *poset*.

The "partial" in partial order comes from the fact that our definition does not require these orders to be total.

Examples:

- $(\mathbb{N}, \leq)$, $(\mathbb{Z}, \leq)$, and $(\mathbb{R}, \leq)$, where $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{R}$ are the sets of natural numbers, integers, and real numbers, respectively, and $\leq$ denotes the usual ordering on these sets. These are all total orders.
- $(S, =)$, where S is any set. All distinct pairs of elements are incomparable in this order. Any partial order of this form in which the order relation contains only the reflexive pairs (x, x) is called a *discrete* partial order.
- $(2^S, \subseteq)$. Here 2^S denotes the *powerset* of S, or the set of all subsets of S, often written $\mathcal{P}(S)$. This is not a total order if S contains more than one element. For example, in $(2^{\{a,b\}}, \subseteq)$, the elements $\{a\}$ and $\{b\}$ are incomparable: neither $\{a\} \subseteq \{b\}$ nor $\{b\} \subseteq \{a\}$.
- $(2^S, \supseteq)$. In fact, if $(S, \sqsubseteq)$ is a partial order, then so is $(S, \sqsupseteq)$, where $s \sqsupseteq t \triangleq t \sqsubseteq s$.
- $(\mathbb{N}, |)$, where $\mathbb{N} = \{0, 1, 2 \ldots\}$ and $a \mid b$ if a divides b; that is, if $b = ka$ for some $k \in \mathbb{N}$. Note that for any $n \in \mathbb{N}$, we have $n \mid 0$; we call 0 an *upper bound* for $\mathbb{N}$ (but only in this ordering, of course!).
- $(\mathbb{Z}, <)$ is not a partial order, because $<$ is not reflexive.
- $(\mathbb{Z}, \sqsubseteq)$, where $m \sqsubseteq n \overset{\triangle}{\Leftrightarrow} |m| \leq |n|$, is not a partial order because $\sqsubseteq$ is not antisymmetric: $-1 \sqsubseteq 1$ and $1 \sqsubseteq -1$, but $-1 \neq 1$.
- $(\mathbb{C}, \sqsubseteq)$, where $\mathbb{C}$ is the set of complex numbers and $x \sqsubseteq y$ if $\|x\| \leq \|y\|$, is not a partial order because $\sqsubseteq$ is not antisymmetric: $i \sqsubseteq 1$ and $1 \sqsubseteq i$, but $i \neq 1$.
- Let S be a set and let $\equiv_1$ and $\equiv_2$ be equivalence relations on S. We say that $\equiv_1$ *refines* $\equiv_2$ if for all $x, y \in S$, if $x \equiv_1 y$, then $x \equiv_2 y$. The relation *refines* is a partial order on the set of all equivalence relations on S. Considering equivalence relations as sets of order pairs, this is just the subset order on $2^{S \times S}$ restricted to equivalence relations.

1.1 Monotone Maps

Let X and Y be posets (we use $\sqsubseteq$ to denote the partial order in both X and Y). A function $f : X \to Y$ is called *monotone* if for all $x, y \in X$, if $x \sqsubseteq y$ in X, then $f(x) \sqsubseteq f(y)$ in Y. In other words, f is monotone if it preserves order. For example, the exponential function $\lambda x.\, e^x : \mathbb{R} \to \mathbb{R}$ is monotone with respect to the natural order $\leq$ on $\mathbb{R}$.

1.2 Hasse Diagrams

Partial orders can sometimes be described pictorially using *Hasse diagrams.*[1] In a Hasse diagram, each element of the partial order is displayed as a (possibly labeled) point, and lines are drawn between these points, according to these rules:

- If x and y are elements of the partial order, and $x \sqsubseteq y$, then the point corresponding to x is drawn lower in the diagram than the point corresponding to y.
- A line is drawn between the points representing x and y iff $x \sqsubseteq y$ and there does not exist a z strictly between x and y in the partial order; that is, the ordering relation between x and y is not due to transitivity.

Here is an example of a Hasse diagram for the subset relation on the set $2^{\{a,b,c\}}$:

$$
\begin{tikzcd}
& \{a,b,c\} \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
\{a,b\} \arrow[d, dash] \arrow[dr, dash] & \{a,c\} \arrow[dl, dash] \arrow[dr, dash] & \{b,c\} \arrow[dl, dash] \arrow[d, dash] \\
\{a\} \arrow[dr, dash] & \{b\} \arrow[d, dash] & \{c\} \arrow[dl, dash] \\
& \varnothing &
\end{tikzcd}
$$

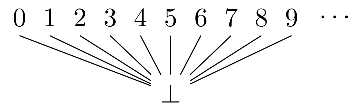
2 Pointed Posets

Given any poset $(S, \sqsubseteq)$, we can add a new bottom element $\bot$ to get a new poset $(S_\bot, \sqsubseteq_\bot)$. We extend $\sqsubseteq$ to make $\bot$ less than everything else, and keep all other relationships the same. Thus we define $S_\bot = S \cup \{\bot\}$, $d_1 \sqsubseteq_\bot d_2$ if $d_1, d_2 \in S$ and $d_1 \sqsubseteq d_2$, and $\bot \sqsubseteq_\bot d$ for all $d \in S_\bot$. Thus $S_\bot$ is the set S with a new least element $\bot$ added below everything in S.

In our semantic domains, we can think of $\sqsubseteq$ as "less information than". Thus nontermination $\bot$ contains less information than any element of S.

Recall that a *discrete partial order* is a poset in which no two distinct elements of S are $\sqsubseteq$-comparable. If we apply this construction to a discrete partial order, we get a *flat partial order*. The only $\sqsubseteq$-relationships among distinct elements are between $\bot$ and every other element. For example, applied to $\mathbb{N}$, we get $\mathbb{N}_\bot$.

[1]Named after Helmut Hasse, 1898–1979. Hasse published fundamental results in algebraic number theory, including the Hasse (or "local-global") principle. He succeeded Hilbert and Weyl as the chair of the Mathematical Institute at Göttingen.



A partial order is called *pointed* if it has a distinguished least element $\perp$. All such lifted partial orders, including flat partial orders, are pointed.

3 Chain-Complete Partial Orders and Continuous Functions

Let $(X, \sqsubseteq)$ be a poset. If $A \subseteq X$, we say that x is an *upper bound* for A if $y \sqsubseteq x$ for all $y \in A$. We say that x is a *least upper bound* or *supremum* of A if

- x is an upper bound for A, and
- for all other upper bounds y of A, $x \sqsubseteq y$.

Upper bounds and suprema need not exist. For example, the set of natural numbers $\mathbb{N}$ under its natural order $\leq$ has no supremum in $\mathbb{N}$. However, if the supremum of any set exists, it is unique. A partially ordered set is said to be *complete* if all subsets have suprema. The supremum of a set C, if it exists, is denoted $\bigsqcup C$.

Note that all elements of X are (vacuously) upper bounds of the empty set $\varnothing$, so if the supremum of $\varnothing$ exists, then it is necessarily the least element of the entire set. In this case we give it the name $\perp$.

A *chain* is a subset of X that is totally ordered by $\sqsubseteq$. For example, in the partial order of subsets of $\{0, 1, 2\}$ ordered by set inclusion, the set $\{\varnothing, \{2\}, \{1, 2\}, \{0, 1, 2\}\}$ is a chain. A partially ordered set is *chain-complete* if all nonempty chains have suprema. A chain-complete partially ordered set is called a CPO. The empty chain $\varnothing$ is not included in the definition of chain-complete, but if the empty chain also has a supremum, then it is necessarily the least element $\perp$ of the CPO.

Let X and Y be CPOs (we use $\sqsubseteq$ to denote the partial order in both X and Y). Recall that a function $f : X \to Y$ is *monotone* if f preserves order; that is, for all $x, y \in X$, if $x \sqsubseteq y$ then $f(x) \sqsubseteq f(y)$. A function $f : X \to Y$ is *continuous* if f preserves suprema of nonempty chains; that is, if $C \subseteq X$ is a nonempty chain in X, then $\bigsqcup_{x \in C} f(x)$ exists and equals $f(\bigsqcup C)$. Here $\bigsqcup_{x \in C} f(x)$ is alternate notation for $\bigsqcup \{f(x) \mid x \in C\}$.

Every continuous map is monotone: if $x \sqsubseteq y$, then $y = \bigsqcup\{x, y\}$, so by continuity $f(y) = f(\bigsqcup\{x, y\}) = \bigsqcup \{f(x), f(y)\}$, which implies that $f(x) \sqsubseteq f(y)$.

In the definition of continuity, we excluded the empty chain $\varnothing$. If it were included, then a continuous function would have to preserve $\perp$; that is, $f(\perp) = \perp$. A continuous function that satisfies this property is called *strict*. We do not include $\varnothing$ in the definition of continuous functions, because we wish to consider non-strict functions, such as the $\mathcal{F}$ of Lecture 20.

The space of continuous functions $D \to E$ is denoted $[D \to E]$.

4 The Knaster–Tarski Theorem in CPOs

Let $F : D \to D$ be any continuous function on a pointed CPO D. Then F has a least fixpoint $\mathit{fix}\, F \triangleq \bigsqcup_n F^n(\perp)$. The proof is a direct generalization of the proof for set operators given in an earlier lecture, where $\perp$ was $\varnothing$ and $\bigsqcup$ was $\bigcup$. In a nutshell: by monotonicity, the $F^n(\perp)$ form a chain; since D is a CPO, the supremum $\mathit{fix}\, F$ of this chain exists; and by continuity, $\mathit{fix}\, F$ is preserved by F.

5 Continuous Functions on CPOs Form a CPO

Now we claim that if C and D are CPOs, then the space of continuous functions $[C \to D]$ is a CPO under the pointwise ordering

$$f \sqsubseteq g \quad \stackrel{\triangle}{\Longleftrightarrow} \quad \forall x \in C\ f(x) \sqsubseteq g(x).$$

It is easily verified that $\sqsubseteq$ is a partial order on $[C \to D \to]$. If D is pointed with bottom element $\bot$, then $[C \to D \to]$ is also pointed with bottom element $\bot \triangleq \lambda x \in C\,.\,\bot$.

We need to show that $[C \to D \to]$ is chain-complete. Let $\mathcal{C}$ be a nonempty chain in $[C \to D \to]$. Define

$$G \quad \triangleq \quad \lambda x \in C\,.\, \bigsqcup_{g \in \mathcal{C}} g(x).$$

First, G is a well-defined function, since for any $x \in C$, $\{g(x) \mid g \in \mathcal{C}\}$ is a chain in D, therefore its supremum $\bigsqcup_{g \in \mathcal{C}} g(x)$ exists. Also, the function G is continuous, since for any nonempty chain E in C,

$$\begin{aligned}
G(\bigsqcup E) &= \bigsqcup_{g \in \mathcal{C}} g(\bigsqcup E) && \text{by the definition of } G \\
&= \bigsqcup_{g \in \mathcal{C}} \bigsqcup_{x \in E} g(x) && \text{since each } g \in \mathcal{C} \text{ is continuous} \\
&= \bigsqcup_{x \in E} \bigsqcup_{g \in \mathcal{C}} g(x) && \text{by the lemma below} \\
&= \bigsqcup_{x \in E} G(x) && \text{again by the definition of } G.
\end{aligned}$$

The third step in the above argument uses the following lemma.

Lemma 23.1. *If a_{xy} is a doubly-indexed collection of members of a partially ordered set such that*

(i) *for all x, $\bigsqcup_y a_{xy}$ exists,*

(ii) *for all y, $\bigsqcup_x a_{xy}$ exists, and*

(iii) *$\bigsqcup_y \bigsqcup_x a_{xy}$ exists,*

then $\bigsqcup_x \bigsqcup_y a_{xy}$ exists and is equal to $\bigsqcup_y \bigsqcup_x a_{xy}$.

Proof. Clearly $\bigsqcup_y \bigsqcup_x a_{xy}$ is an upper bound for all a_{xy}, therefore it is an upper bound for all $\bigsqcup_y a_{xy}$; and if b is any other upper bound for all $\bigsqcup_y a_{xy}$, then $a_{xy} \sqsubseteq b$ for all x, y, therefore $\bigsqcup_y \bigsqcup_x a_{xy} \sqsubseteq b$, so $\bigsqcup_y \bigsqcup_x a_{xy}$ is the least upper bound for all $\bigsqcup_y a_{xy}$; that is, $\bigsqcup_x \bigsqcup_y a_{xy} = \bigsqcup_y \bigsqcup_x a_{xy}$. $\square$

To apply this lemma, we need to know that

(i) for all $g \in \mathcal{C}$, $\bigsqcup_{x \in E} g(x)$ exists,

(ii) for all $x \in E$, $\bigsqcup_{g \in \mathcal{C}} g(x)$ exists, and

(iii) $\bigsqcup_{g \in \mathcal{C}} \bigsqcup_{x \in E} g(x)$ exists.

But (i) holds because all $g \in \mathcal{C}$ are continuous, therefore $\bigsqcup_{x \in E} g(x) = g(\bigsqcup E)$; (ii) holds because $\{g(x) \mid g \in \mathcal{C}\}$ is a chain in D, and D is chain-complete; and (iii) follows from (i) and (ii) by taking $x = \bigsqcup E$.

6 Fixpoints and the Semantics of while-do

Now let us return to the denotational semantics of the while loop. We previously defined the function

$$\begin{array}{rcl} \mathcal{F} & : & (\mathit{Env} \to \mathit{Env}_\perp) \;\to\; (\mathit{Env} \to \mathit{Env}_\perp) \\ \mathcal{F} & \triangleq & \lambda w \in \mathit{Env} \to \mathit{Env}_\perp .\, \lambda\sigma \in \mathit{Env}.\, \mathit{if}\ \mathcal{B}[\![b]\!]\sigma\ \mathit{then}\ w^\dagger(\mathcal{C}[\![c]\!]\sigma)\ \mathit{else}\ \sigma. \end{array}$$

Any function $\mathit{Env} \to \mathit{Env}_\perp$ is continuous, since chains in the discrete space Env contain at most one element, thus the space of functions $\mathit{Env} \to \mathit{Env}_\perp$ is the same as the space of continuous functions $[\mathit{Env} \to \mathit{Env}_\perp \to]$. Moreover, the lift $w^\dagger : \mathit{Env}_\perp \to \mathit{Env}_\perp$ of any function $w : \mathit{Env} \to \mathit{Env}_\perp$ is continuous.

By previous arguments, the function space $[\mathit{Env} \to \mathit{Env}_\perp \to]$ is a pointed CPO, and $\mathcal{F}$ maps this space to itself. To obtain a least fixpoint by Knaster–Tarski, we need to know that $\mathcal{F}$ is continuous.

Let us first check that it is monotone. This will ensure that, when trying to check the definition of continuity, when C is a chain, $\{\mathcal{F}(d) \mid d \in C\}$ is also a chain, so that $\bigsqcup_{d\in C} \mathcal{F}(d)$ exists. Suppose $d \sqsubseteq d'$. We want to show that $F(d) \sqsubseteq F(d')$. But for all σ,

$$\begin{array}{rcl} \mathcal{F}(d)(\sigma) & = & \mathit{if}\ \mathcal{B}[\![b]\!]\sigma\ \mathit{then}\ d^\dagger(\mathcal{C}[\![c]\!]\sigma)\ \mathit{else}\ \sigma \\ & \sqsubseteq & \mathit{if}\ \mathcal{B}[\![b]\!]\sigma\ \mathit{then}\ (d')^\dagger(\mathcal{C}[\![c]\!]\sigma)\ \mathit{else}\ \sigma \\ & = & \mathcal{F}(d')(\sigma). \end{array}$$

Here we have used the fact that the operator $(\cdot)^\dagger$ is monotone, which is easy to check.

Now let us check that $\mathcal{F}$ is continuous. Let C be an arbitrary chain. We want to show that $\bigsqcup_{d\in C} \mathcal{F}(d) = \mathcal{F}(\bigsqcup C)$. We have

$$\begin{array}{rcl} \displaystyle\bigsqcup_{d\in C} \mathcal{F}(d) & = & \displaystyle\bigsqcup_{d\in C} \lambda\sigma.\, \mathit{if}\ \mathcal{B}[\![b]\!]\sigma\ \mathit{then}\ d^\dagger(\mathcal{C}[\![c]\!]\sigma)\ \mathit{else}\ \sigma \\ & = & \displaystyle\lambda\sigma.\, \bigsqcup_{d\in C} \mathit{if}\ \mathcal{B}[\![b]\!]\sigma\ \mathit{then}\ d^\dagger(\mathcal{C}[\![c]\!]\sigma)\ \mathit{else}\ \sigma \\ & = & \displaystyle\lambda\sigma.\, \mathit{if}\ \mathcal{B}[\![b]\!]\sigma\ \mathit{then}\ \bigsqcup_{d\in C} d^\dagger(\mathcal{C}[\![c]\!]\sigma)\ \mathit{else}\ \sigma \\ & = & \displaystyle\lambda\sigma.\, \mathit{if}\ \mathcal{B}[\![b]\!]\sigma\ \mathit{then}\ (\bigsqcup C)^\dagger(\mathcal{C}[\![c]\!]\sigma)\ \mathit{else}\ \sigma \quad = \quad \mathcal{F}(\bigsqcup C), \end{array}$$

since $\mathcal{B}[\![b]\!]\sigma$ does not depend on d and since the lift operator $(\cdot)^\dagger$ is continuous.