Lecture 23

Topics

1. If you can attend Voevodsky’s lecture at 4:30 pm in Malott 406 it will be relevant to discussion in Friday’s lecture.

2. To reason about partial functions we need a new equality relation, \( t_1 \simeq t_2 \), and a new induction principle, fixed point induction. We first discuss the fixed points of recursive functionals.

3. We will study Kleene’s Recursion Theorem as a basis for the induction.

4. We will study a generalization of Kleene’s theorem in a classical setting.

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Kleene Equality \( \varphi(x) \simeq \psi(x) \) – converge or diverge together and if they converge, then they converge to the same value.

Functionals and fixed points

Consider the recursive function on \( \mathbb{N} \):
\[
f(x, y) = \text{if } x = y \text{ then } y + 1 \text{ else } f(x, f(x-1, y + 1)).
\]

Write this in terms of the functional \( F \):
\[
F(f) = \lambda x. \lambda y. \text{if } x = y \text{ then } y + 1 \text{ else } f(x, f(x-1, y + 1)).
\]

Let
\[
\begin{align*}
  f_1(x, y) &= \text{if } x = y \text{ then } y + 1 \text{ else } x + 1 \\
  f_2(x, y) &= \text{if } x \geq y \text{ then } x + 1 \text{ else } y - 1 \\
  f_3(x, y) &= \text{if } x \geq y \& \text{even}(x - y) \text{ then } x + 1 \text{ else } \bot (\text{where } \bot \text{ is the diverging element})
\end{align*}
\]

Notice that for \( i = 1, 2, 3 \)
\[
F(f_i)(x, y) \simeq \text{if } x = y \text{ then } y + 1 \text{ else } f_i(x, f_i(x-1, y + 1))
\]
Notice that \( f_3 \) is a fixed point of \( F \), i.e.
\[
F(f_3) \simeq f_3
\]
\[
F(f_3) = \lambda x, y. \text{if } x = y \text{ then } y + 1 \text{ else } f_3(x, f_3(x - 1, y + 1))
\]
\[
= \lambda x, y. \text{if } x = y \text{ then } y + 1 \text{ else if } x \geq y \text{ and even}(x - y) \text{ then } x + 1 \text{ else } \bot
\]
\[
= \text{if } x = y \text{ then even}(x - y)
\]
\[
\text{hence } x + 1 \text{ (same value as } f_3)\]
\[
\text{if } x \neq y \text{ then }
\]
\[
\text{so if even}(x - y) \text{ then } x + 1 \text{ (same value as } f_3)
\]
\[
\text{if } x < y \text{ then } \bot \text{ (same value as } f_3)\]

But also notice:
\[
F(f_1) \simeq f_1 \text{ since}
\]
\[
F(f_1) = \lambda x, y. \text{if } x = y \text{ then } y + 1
\]
\[
\text{so if } x = y \text{ this is the same value as } f_1
\]
\[
\text{if } x \neq y \text{ then } f_1(x, f_1(x - 1, y + 1)) \text{ and } f_1(x, \ldots) \text{ is } x + 1, \text{ this is the same value as } f_1
\]

Notice \( f_3(x, y) \sqsubseteq f_1(x, y) \).

**Kleene’s Recursion Theorem** For all recursive functionals \( F(\varphi) \simeq \varphi' \) there is a partial recursive function \( \varphi \) such that \( F(\varphi) \simeq \varphi \), and for all \( \varphi' \) such that \( F(\varphi') \simeq \varphi' \), \( \varphi \sqsubseteq \varphi' \).

**Proof sketch:**

Let \( \varphi_0 = \) the totally undefined partial function on \( \mathbb{N} \). Define the sequence \( \varphi_{i+1} \simeq F(\varphi_i) \), note \( \varphi_i \sqsubseteq \varphi_j, i < j \).

Define \( \varphi_\omega \) as the limit of this sequence. \( \varphi_\omega(x) \) is defined if there is an \( i \) such that \( \varphi_i(x) \downarrow \). To compute \( \varphi_\omega(x) \) we compute the sequence \( \varphi_1(x), \varphi_2(x), \ldots, \varphi_n(x) \) and give a value if one of the \( \varphi_j(x) \) is defined.

We need to establish two claims:

(a) \( F(\varphi_\omega)(x) \simeq \varphi_\omega(x) \) for all \( x \)

(b) If \( F(\varphi)(x) \simeq \varphi(x) \) for all \( x \), then \( \varphi_\omega(x) \sqsubseteq \varphi(x) \)

Why are these intuitively true?

(a) \( F(\varphi_\omega)(x) \simeq \varphi_\omega(x) \) because if \( F(\varphi_\omega)(x) = k \) for some number \( k \) then at some least \( i, F(\varphi_i(x)) = k \). \( F(\varphi_\omega)(x) \) will give the same value because it accesses the same data.

(b) If \( F(\varphi)(x) \simeq \varphi(x) \), then \( \varphi_\omega(x) = \varphi(x) \) because \( \varphi_0 \sqsubseteq \varphi, \varphi_1 \sqsubseteq \varphi, \ldots, \varphi_i \sqsubseteq \varphi \), and this sequence has the least amount of data needed to compute the fixed point.