Expected Utility Theory

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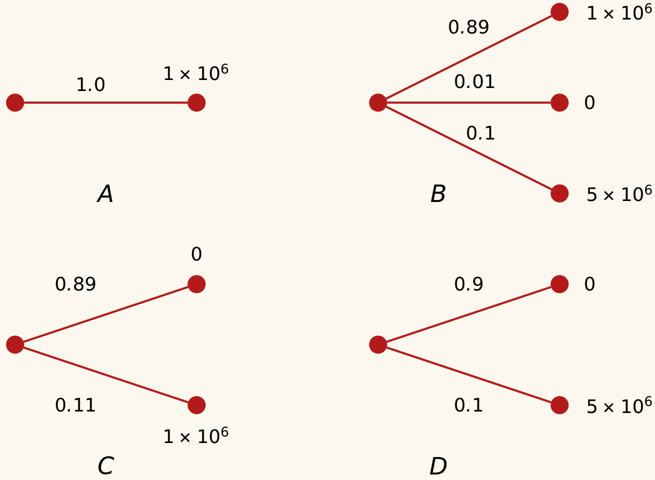


THE LOTTERY
masslottery.com
POWERBALL
POWERPLAY
DATE August 23, 2017
PAY TO THE ORDER OF:
Powerball Winner!
$758,700,000
SEVEN HUNDRED FIFTY-EIGHT MILLION, SEVEN HUNDRED THOUSAND!!
DOLLARS
MEMO
CONGRATULATIONS!

Powerball Payouts

Prize Level	Payout	Odds
Match 5 + PB	Jackpot	1 in 292,201,338
Match 5	$1,000,000	1 in 1,688,054
Match 4 + PB	$50,000	1 in 913,129
Match 4	$100	1 in 36,525
Match 3 + PB	$100	1 in 14,494
Match 3	$7	1 in 580
Match 2 + PB	$7	1 in 701
Match 1 + PB	$4	1 in 92
Match 0 + PB	$4	1 in 38

Source: http://www.powerball.net

The Jackpot is currently estimated at $90 mil. — $68 mil. cash value

Payout	Probability
68×10^6	3.4223×10^{-9}
1×10^6	5.99501×10^{-7}
$50,000$	1.09514×10^{-6}
100	9.63726×10^{-5}
7	3.15067×10^{-3}
4	3.71854×10^{-2}

Expected winnings are $1.0603. A ticket costs $2, so the net expected gain is −$0.9397.

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For a net expected payoff of 0 the cash payout must be $342.6 mil.

An Important Fact!

Deductions for Gambling Losses

Playing the lottery is classed as gambling as far as the Internal Revenue Service (IRS) is concerned, which means that you are entitled to a tax deduction on any losses incurred. To file these deductions, you will need to keep an accurate record of your wins and losses, as well as any evidence of them, such as the tickets you bought. You must itemize the deductions on the tax form 1040, obtainable from the IRS website. The losses you deduct cannot exceed your income from all forms of gambling, including but not limited to horse racing, casinos, and raffles.

Source https://www.powerball.net/taxes

Problems with Expected Values

Am I 17 million times better off winning $68 million than I am winning $4?.

Problems with Expected Values

Pascal's Wager

	God Exists	There is No God
Succeed in Believing	Eternal Life	Finite, Deluded Life
Remain an Atheist	Oh, Hell!	What I now presume

Problems with Expected Values

Pascal's Wager

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	God Exists	There is No God
Succeed in Believing	∞	x
Remain an Atheist	y	z

At what odds should you choose to remain an atheist?

Problems with Expected Values

The Saint Petersburg Paradox

Here is a game:

- Flip a fair coin until tails comes up.
- If the first T appears at flip N, you are paid $\$2^N$.

How much would you be willing to pay to play this game?

Problems with Expected Values

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$$\mathrm{E}\left\{2^{\tilde{N}}\right\} = \frac{1}{2}\cdot 2 + \frac{1}{4}\cdot 4 + \cdots + \frac{1}{2^n}\cdot 2^n + \cdots = +\infty$$

History of EU

- Daniel Bernoulli's resolution of the St. Petersburg paradox was to introduce expected utility. He observed that although $\sum_n 2^{-n}2^n$ is unbounded, $\sum_n 2^{-n} \log 2^n = 2 \log 2$. If you were to pay w for the wager, your expected utility is $\sum_n 2^{-n} \log(2^n - w)$. This is a good bet for $w < \$1.81$.
- It came under criticism in the 1930s.
  - People evaluate gambles by looking at the mean, the variance, and other statistics. (Hicks 1931)
  - The utility function whose expectation is being taken is "cardinal". (Tintner 1942)

So the late 40's and early 50's brought forth the zoo of functions.

History of EU

- von Neumann and Morgenstern (1947) provided the first axiomatization of EU preferences.

- Much of the acceptance of EU in the early 50s came from the normative force of the axioms. But there has been pushback against the descriptive validity of EU since the late 40s. Milton Friedman and Leonard Savage wrote a famous and famously bad paper in 1948 that tried to reconcile the fact that people buy both insurance and lottery tickets with EU. Few of the major players in 1950s decision theory thought that EU was empirically valid.

What’s To Like About EU

- Additive separability across states - independence axiom, sure thing principle
- Stochastic monotonicity
- Representation properties e.g. risk aversion

Characterization of EU Preferences

When does a preference relation have an EU representation?

- X Outcomes of lotteries (finite for now)
- $\mathscr{L}$ $\ell = \{p_1, x_1; \ldots, p_N, x_N)\}$ A simple lottery with probabilities $p_1, \ldots, p_N$ and prizes $x_1, \ldots, x_N$.
- Δ is the set $\left\{(p_1, \ldots, p_N) : p_n \geq 0 \quad \& \quad \sum_n p_n = 1\right\}$.

Characterization of Preferences
Notation

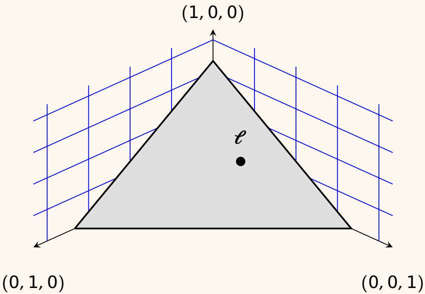
Kreps provides three levels of organization

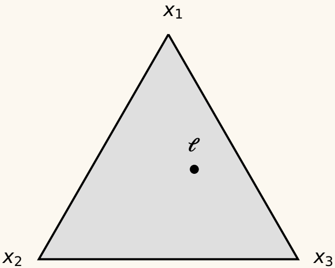
- A fixed & finite prize space. Lotteries are elements in Δ. Useful for drawing pictures.

- An infinite set of possible prizes, but lotteries are probability distributions with **finite support**, that is, each lottery assigns probability 1 to some finite set of prizes. This is what Kreps means by a "simple lottery". The set of such lotteries is a mixture space. Kreps calls this P_S but since P is overloaded in this class I will call it $\mathscr{L}$.
- Lotteries with countable and continuum support. Kreps doesn't name it and neither will I.

Three Alternatives

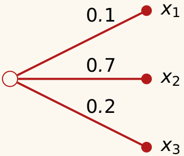
$$\ell = (0.4, x_1;\ 0.2, x_2;\ 0.4, x_3)$$



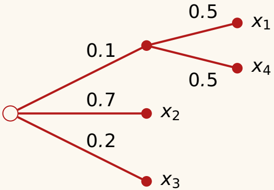


A representation of Δ and $\mathscr{L}$.

Simple & Compound Lotteries

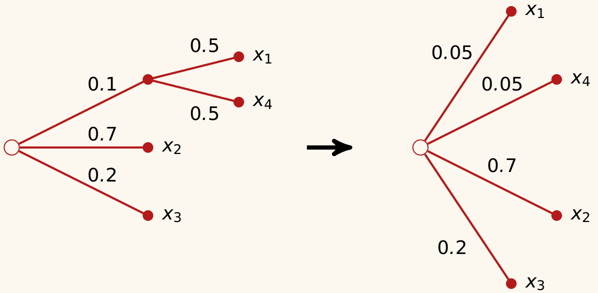


A Simple Lottery



A Compound Lottery

Reduction of Compound Lotteries



Mixture Spaces

Definition. A mixture space is a non-empty set $\mathscr{M}$ together with an operation

$$[0,1] \times \mathscr{M} \times \mathscr{M} \to \mathscr{M}$$

$$(\lambda, \boldsymbol{\ell}, \boldsymbol{m}) \to \boldsymbol{\ell}\lambda\boldsymbol{m}$$

s.t.

- A.1 $\boldsymbol{\ell}1\boldsymbol{m} = \boldsymbol{\ell}$,
- A.2 $\boldsymbol{\ell}\lambda\boldsymbol{m} = \boldsymbol{m}(1-\lambda)\boldsymbol{\ell}$,
- A.3 $(\boldsymbol{\ell}\lambda\boldsymbol{m})\mu\boldsymbol{m} = \boldsymbol{\ell}(\lambda\mu)\boldsymbol{m}$.

A function $U : \mathscr{M} \to \mathbb{R}$ is mixture-preserving if for all $\lambda, \boldsymbol{\ell}, \boldsymbol{m}$

$$U(\boldsymbol{\ell}\lambda\boldsymbol{m}) = \lambda U(\boldsymbol{\ell}) + (1-\lambda)U(\boldsymbol{m})$$

Some Properties of Mixture Spaces

- $\ell 0 m = m$
  Proof. $\ell 0 m \overset{A2}{=} m 1 \ell \overset{A1}{=} m$.
- $\ell \lambda \ell = \ell$
  Proof. $\ell \lambda \ell \overset{A1}{=} (\ell 1 \ell)\lambda \ell \overset{A2}{=} (\ell 0 \ell)\lambda \ell \overset{A3}{=} \ell 0 \ell = \ell$.
- $(\ell \lambda m)\mu(\ell \nu m) = \ell(\lambda\nu + (1-\lambda)\nu)m$
  Proof. See Fishburn (1982).

Examples of Mixture Spaces

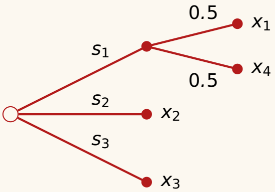
- A convex set is a mixture space with $\ell\lambda m = \lambda\ell + (1-\lambda)m$.
- $\mathscr{M} = \{\ell, m, n\}$. For all λ, $m\lambda m = m$ and $n\lambda n = n$, $\ell 0 m = m 1 \ell = n 0 m = m 1 n = m$, $\ell 0 n = n 1 \ell = m 0 n = n 1 m = n$, and all other mixtures equal ℓ.
- $\mathscr{M} = \{\ell, m, n\}$. For all $\lambda \in (0,1)$, $\ell\lambda m = m\lambda\ell = m$, $m\lambda n = n\lambda m = n$, $n\lambda\ell = \ell\lambda n = \ell$, and the 0,1 mixtures are chosen according to A.1 and A.2.

Here are two properties of convex sets not shared by all mixture spaces:

- $(\ell\beta m)\alpha n = \ell\alpha\beta(q\alpha(1-\beta)(1-\alpha\beta)^{-1}n)$ (associativity)
- for $\alpha \neq 0$, $\ell\alpha n = m\alpha n$ implies that $\ell = m$. (determinacy)

Why Mixture Spaces?

Why not just reduce compound lotteries to simple lotteries and embed them in a convex set as the picture on slide 14 suggests?



Reduce this!

s_1, s_2 and s_3 are states of nature without given probabilities.

The Mixture Space Representation Theorem

Theorem. Let $\succ$ be a binary relation on a mixture space $\mathscr{M}$. Assume:

- (Preference) $\succ$ is a preference relation,
- (Archimedean) For all $\ell, m, n \in \mathscr{M}$ such that $\ell \succ m \succ n$, there are $0 < \alpha, \beta < 1$ such that

$$\ell \alpha n \succ m \succ \ell \beta n,$$

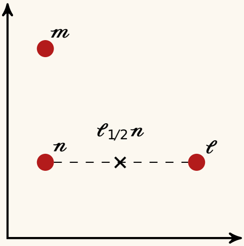
- (Independence) For all $\ell, m, n \in \mathscr{M}$ and $0 < \alpha \leq 1$, if $\ell \succ m$ then

$$\ell \alpha n \succ m \alpha n.$$

Then $\succ$ has a mixture-preserving representation: $m \succ n$ iff $U(m) > U(n)$, and $U(m \alpha n) = \alpha U(m) + (1-\alpha)U(n)$. Furthermore, if V is another mixture-preserving representation, for some $\alpha > 0$ and β,

$$V(m) = \alpha U(m) + \beta.$$

Archimedean Axiom

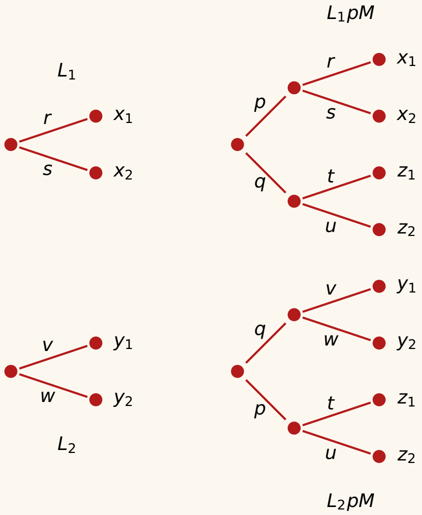


The Lexicographic Order

- $\ell \succ m \succ n$.
- For all $\alpha > 0$, $\ell \alpha n \succ m$

Independence Axiom

The usual justification goes as follows:



Independence: $L_1 \succ L_2$ iff $L_1pM \succ L_2pM$.

Suppose L_1 is preferred to L_2. Now imagine flipping a coin and getting L_1 on H and a default lottery M on T. How should it compare to getting L_2 on H and the same default on T.

This sounds normatively plausible. Descriptively the reduction of compound lotteries is questionable. The Allais paradox (to come) provides a counterexample.

Proof of the Representation Theorem

Here are a bunch of facts that can be quickly derived:

- If $m \succ n$ and $0 \geq \alpha < \beta \geq 1$ then $m\beta n \succ m\alpha n$.

  This is a monotonicity property.

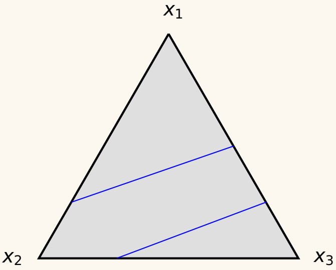
- If $\ell \succsim m \succsim n$ and $\ell \succ n$, then for exactly one $0 \leq \alpha \leq 1$ $m \sim \ell\alpha n$.

  Among other things, this suggests that indifference curves are not thick.

- If $\ell \sim m$ then $\ell\alpha n \sim m\alpha n$

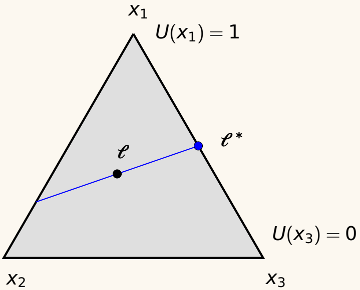
  This says two things:

  - If $\ell \sim m$ then for all $0 \leq \alpha \leq 1$ $\ell\alpha m \sim m\alpha m = m$, Indifference curves are lines.
  - Indifference curves are parallel.



Proof of the Representation Theorem

Idea. Suppose x_1 is the best mixture and x_3 is the worst.



$U(\ell)$ solves

$$\ell \sim \ell^*$$

and

$$\ell^* = x_1 U(\ell) x_3$$

This idea extends to the case where there is no best and worst mixture. See Kreps p. 55.

Application to von Neumann–Morgenstern EU

Theorem. Let $\mathscr{L}$ denote the set of lotteries on a finite outcome space X and let $\succ$ be a preference order satisfying axioms A.1.–3. Then there exists a $u : X \to \mathbb{R}$ such that

$$U(p) = E_p u \equiv \sum_n p_n u(x_n)$$

represents $\succ$. Furthermore, $v : X \to \mathbb{R}$ similarly represents $\succ$ iff $v = \alpha u + \beta$ with $\alpha > 0$.

Proof. $\mathscr{L}$ with the convex combination operation is a mixture space. The mixture space representation theorem gives a representation $U : \ell \to \mathbb{R}$ such that $U(p\gamma q) = \gamma U(p) + (1-\gamma)U(q)$. Arguing by induction on the cardinality of X proves that $U(p) = \sum_n p_n u(x_n)$. ■

Cardinal Utility?

A relational system $\mathscr{R} = \langle X, R_1, \ldots, R_K \rangle$ is a set X of objects together with K relations. These relations may be binary, ternary, etc.

If, for example, $\mathscr{R} = \langle X, R_1, R_2, R_3 \rangle$ is a relational system, and if R_1 and R_3 are binary while R_2 is ternary, we say that the type of $\mathscr{R}$ is $2, 3, 2$.

A function F is a numerical representation of the relational system $\mathscr{R}$ iff there is a real relational system (on the object set of real numbers) of the same type, $\langle \mathbb{R}, S_1, \ldots, S_K \rangle$ and a function $F : X \to \mathbb{R}$ such that for all k, $(x_1, \ldots, x_{m_k}) \in R_k$ iff $((F(x_1), \ldots, F(x_{m_k})) \in S_k$.

Cardinal Utility?

It is often said that the "utility" u is a cardinal measure of utility on X, because any other "utility" v is related to u by a positive affine transformation.

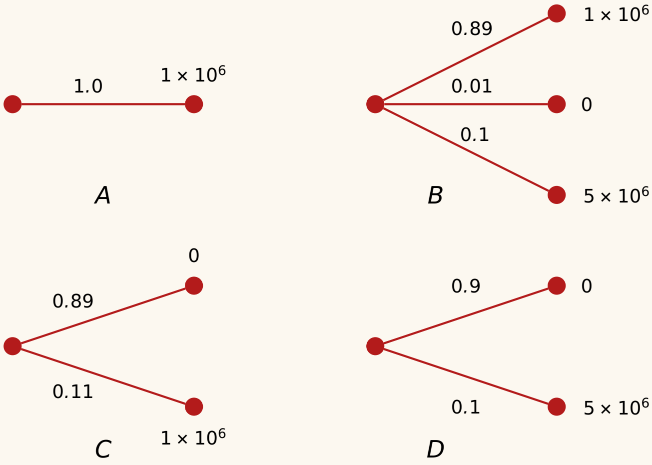
$$u(w) - u(x) > u(y) - u(z) \text{ iff } v(w) - v(x) > v(y) - v(z).$$

Since relations between utility differences are invariant under all "utilities" that appear in vn-M representations of $\succ$, it must be they are significant, that there is an implicit quaternary relationship R' on X, to wit, $(w, x, y, z) \in R'$ iff the decision maker prefers w over x more than she prefers y over z. It is claimed that this is meaningful.

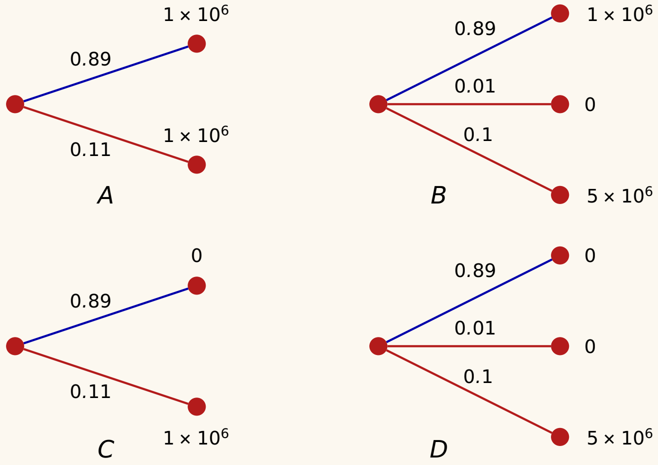
If this were true, then we would have derived the quaternary relationship "more better than" from the binary relationship "better than".

What's wrong with this?

The Allais Paradox



The Allais Paradox



Calling Out BS

Consider the following mixture space:

- O is a finite set of outcomes.
- X_0 is the set of sure prizes.
- X_n consists of all **binary** lotteries on X_{n-1}, $(\lambda, x; (1-\lambda), y)$ where $0 \leq \lambda \leq 1$ and $x, y \in X_{n-1}$.
- $X = \cup_n X_n$.

Identify the elements $(1, x; 0, y)$ and x. Then $X_{n-1} \subset X_n$. This lets us define on X a mixture operator: For $x \in X_m$ and $y \in X_n$ define

$$x\lambda y = (\lambda, x; (1-\lambda), y) \in X_{\max\{m,n\}+1}.$$

If $\succ$ on X satisfies A.1–3 then it has a mixture-preserving representation.

Calling Out BS

For distinct outcomes o_1, o_2, o_3, $o_1\lambda(o_2\gamma o_3) \in X_2$.

$$
\begin{aligned}
U\big(o_1\lambda(o_2\gamma o_3)\big) &= \lambda U(o_1) + (1-\lambda)U(o_2\gamma o_3) \\
&= \lambda U(o_1) + (1-\lambda)\big(\gamma U(o_2) + (1-\gamma)U(o_3) \\
&= \lambda U(o_1) + (1-\lambda)\gamma U(o_2) + (1-\lambda)(1-\gamma)U(o_3)
\end{aligned}
$$

but although $\big(\lambda, o_1; (1-\gamma)(\gamma, o_2; (1-\gamma), o_3\big)$ exists in X_2, the lottery $(\lambda, o_1; (1-\lambda)\gamma, o_2; (1-\lambda)(1-\gamma)o_3\big)$ does not exist in X.

EU With Monetary Prizes

Definition. A decision maker is risk averse if for any gamble $\ell = (p_1, x_1; \ldots, p_N, x_N)$, $(1, \sum_n p_n x_n) \succsim \ell$.

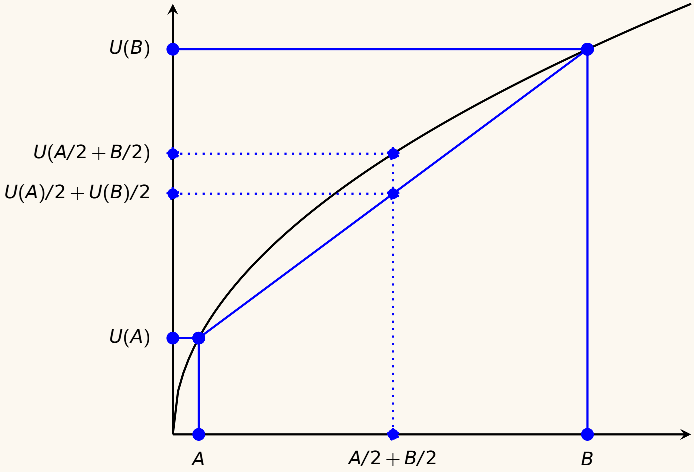
In expected utility terms,

$$U(\mathrm{E}\{X\}) \geq \mathrm{E}\left\{U(X)\right\}$$

This will be true for all gambles iff U is concave.

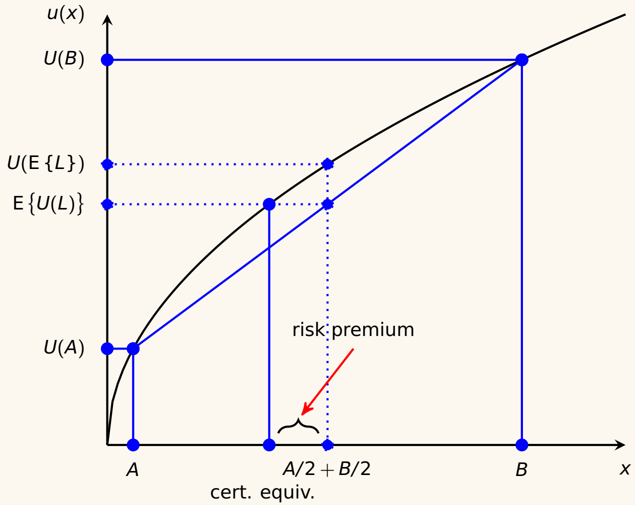
Concave Functions

A concave function's graph is on or above any of its secant lines.

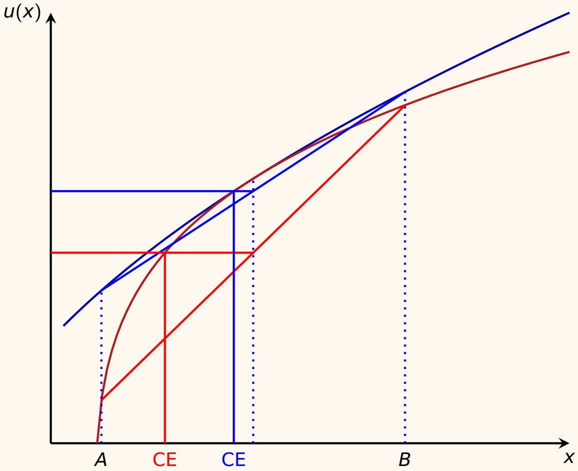


Concave Functions

The lottery $L = (1/2, A; 1/2, B)$.



Curvature and Risk Aversion



Measuring Curvature

Curvatures are measured by coefficients of risk aversion.

- The coefficient of absolute risk aversion is

$$\rho_A(x) = -\frac{u''(x)}{u'(x)} = \frac{d}{dx}\log u'(x)$$

- The coefficient of relative risk aversion is

$$\rho_R(x) = -\frac{xu''(x)}{u'(x)} = \frac{d}{d\log x}\log u'(x)$$

Constant Risk Aversion

Proposition A utility function u has Constant Absolute Risk Aversion $\kappa > 0$ iff

$$u(x) = -e^{-\kappa x}$$

Proposition A utility function u has Constant Relative Risk Aversion $\gamma > 0$. Iff

$$u(x) = \frac{1}{1-\gamma} x^{1-\gamma}$$

or, for $\gamma = 1$,

$$u(x) = \log x.$$

The coefficients are preserved by positive affine transformations.

Risk Aversion

Homework problem: Show that if u is CARA and F is a gamble whose payoff is normally distributed with mean μ and variance σ^2, then

$$CE(F) = \mu - \kappa\sigma^2$$

Proposition. If u is any increasing, concave and C^2 payoff function, and F_σ is a family of gambles with fixed mean μ and variance σ^2, then if σ^2 is sufficiently small,

$$CE(F_\sigma) \approx \mu - \rho_A(\mu)\sigma^2$$

Comparisons of Risk Aversion

What does it mean to say that individual Y is at least as risk averse than is individual Z?

- For any gamble F, $CE^Y(F) \geq CE^Z(F)$,
- For all x, $\rho_A^Y(x) > \rho^Z(x)$.
- There is a concave function $g : \mathbb{R} \rightarrow \mathbb{R}$ such that $u^Y = g \circ u^Z$.

These are all equivalent.

An Application

An individual has an initial wealth w and may lose 1 unit with probability p. She can buy insurance. To insure x dollars of the loss costs $q \cdot x$. Under what circumstances will she buy insurance, and when she buys, how much?

If she buys x dollars of insurance, her expected utility is

$$U(x) = (1-p)u(w-qx) + pu(w-qx-1+x).$$

Actuarially fair insurance requires $q = p$.

Unfair Insurance

Suppose $q > p$. Will she choose $x = 1$?

$$U'(1) = \big(p(1-q) - q(1-p)\big)u'(w-q) = (p-q)u'(w-q) < 0$$

so NO!

Suppose $q = p$. Will she choose $x = 1$?

$$U'(1) = \big(p(1-q) - q(1-p)\big)u'(w-q) = (p-q)u'(w-q) = 0$$

so YES! For any x, $\mathrm{E}\{\tilde{w}\} = w - p$. For $x < 1$,

$$CE(x) < \mathrm{E}\{\tilde{w}\} = w - q = CE(1)$$

so $x = 1$ has a higher certainty equivalence than $x < 1$, hence is optimal.

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