Decision Theory Prelim: Solutions
October 16,2008

There are three questions. Please answer each question in a separate blue book. The test is out of 100; you have 75 minutes. Good luck!

I (30 points) Provide a brief answer for each of the following.

(a) Let $X = \{1, 2, \cdots, N\}$, $N \geq 3$, and suppose that $x \succ y$ if and only if $x > y + 1$. As usual, define $\succeq$ by $x \succeq y$ if *not*$[y \succ x]$. Is this binary relation complete?

Solution: We showed in class that $\succeq$ is complete iff $\succ$ is asymmetric. It is clear that $\succ$ is asymmetric: we can't have $x \succ y$ and $y \succ x$. We accepted that as a full answer, but here are more details: Since $\succ$ is asymmetric, we must have either $\neg(y \succ x)$ or $\neg(x \succ y)$, which means we have either $x \succeq y$ or $y \succeq x$, which is exactly what it takes for $\succeq$ to be complete. (As an aside, $\succ$ is *not* negatively transitive, so $\succeq$ is not transitive.

(b) Let $X = \{a, b, c\}$ and suppose we have a choice function such that $c(\{a, c\}) = \{c\}, c(\{a, b\}) = \{a\}, c(\{b, c\}) = \{c\}, c(\{a, b, c\}) = \{a, c\}$. Is there a preference relation $\succ$ such that $C(\cdot) = C(\cdot, \succ)$?

Solution: No, there is no such preference relation. We showed in class that there is a preference relation iff Sen's α and β held. Since $a \in c(\{a, b, c\})$, Sen's α would require that $a \in c(\{a, b\})$, but it's not. Thus, Sen's α doesn't hold. (As it happens, Sen's β does hold.)

(c) Suppose someone is deciding whether to quit smoking, having heard that smoking might shorten his life span. Of course, he'd prefer to live to a ripe old age, but he enjoys smoking, and if he does make it to old age, he'd prefer to have smoked to not having smoked; likewise if he's going to die early in any case. Thus, he constructs the following decision table, where Q represents quit, C represents continue smoking, L represents live to an old age, and D represents die early.

	L	D
Q	95	-5
C	100	0

He notices that continuing to smoke dominates quitting (no matter whether he lives to a ripe old age or dies early he is better off smoking).

Under what circumstances is this an appropriate representation of the problem; under what circumstances is it not?

Solution: It is an appropriate representation if how long you live is independent of whether you smoke. This may be true, even if smoking and when you die are correlated. For example, you might believe that there is a gene that causes you to smoke and the same gene causes you to be likely to die early. Whether or not you smoke does not affect whether you have the gene. On the other hand, the representation is inappropriate if how long you live does depend on whether you smoke.

Grading: Each part of I was worth 10 points. For (a) several answer books showed that $\succ$ is not transitive. This is true, but it has nothing to do with completeness. This answer received 0 points. On (b) some answer books contained a confusion about whether it was Sen's α or Sen's β that did not hold. If this was only a confusion about the names, and the argument was really about α, then the penalty was 2 points. If there was an argument that β does not hold, and the argument really was about β, the answer received 0 points. Some answer books hypothesized that there was a preference relation and showed that $c(\{a,c\}) = \{c\}$ implies $c \succ a$ and $c(\{a,b,c\}) = \{a,c\}$ implies $\mathit{not}[c \succ a]$. This approach is fine and this answer got full credit. For (c) some answer books changed the payoffs arguing that the decision maker should not like smoking as it harmful to him or to others. If this was the only argument offered the score was 0 as the payoffs represent the decision maker's feelings about smoking and they can be an accurate (from his point of view) representation of how he feels about smoking. If you continued with this argument to note that smoking is harmful because it affects the likelihood of living or dying then the penalty was 3 points, II (30 Points) Consider the following decision matrix:

	s_1	s_2	s_3	s_4
a_1	6	8	9	4
a_2	2	9	9	11
a_3	4	4	4	4
a_4	0	5	15	6

(a) Order the acts according to each of the following decision rules:

  (i) maximin;

  (ii) optimism-pessimism, with index $\alpha = 1/3$;

  (iii) minimax regret;

  (iv) principle of insufficient reason.

(b) Suppose that we multiplied all the utilities by 2 and added 5 (so that, for example, the first row in the matrix would become 16, 21, 21, 13). Which, if any, of the orders in part (a) would change.

(c) [GRAD: for those taking CS5846 or ECON6760 only] Suppose that we squared all the utilities (so that, for example, the first row in the matrix would become 36, 64, 64, 16). Now which of the orders in part (a) would change?

Solution:

(a) (i) $worst(a_1) = 4$; $worst(a_2) = 2$; $worst(a_3) = 4$; $worst(a_4) = 0$. Thus, according to maximin, we have $a_3 \sim a_1 \succ a_2 \succ a_4$.
(ii) $best(a_1) = 9$; $best(a_2) = 11$; $best(a_3) = 4$; $best(a_4) = 15$;
Thus, $opt^{1/3}(a_1) = 1/3 best(a_1) + 2/3 worst(a_1) = 17/3$. Similarly, $opt^{1/3}(a_2) = 15/3 = 5$; $opt^{1/3}(a_3) = 4$; $opt^{1/3}(a_4) = 5$, so $a_1 \succ a_2 \sim a_4 \succ a_3$ according to the optimism-pessimism rule.
(iii) $a_{s_1} = a_1$ (recall that a_{s_1} is the best act if s_1 is the true state); $a_{s_2} = a_2$; $a_{s_3} = a_4$; $a_{s_4} = a_2$. Thus, $regret(a_1) = \max(0, 1, 6, 7) = 7$; $regret(a_2) = \max(4, 0, 6, 0) = 6$; $regret(a_3) = \max(2, 5, 11, 7) = 11$; $regret(a_4) = \max(6, 4, 0, 5) = 6$, so $a_4 \sim a_2 \succ a_1 \succ a_3$ according to the minimax regret rule.
(iv) If Pr is the uniform distribution, $E_{\Pr}(u_{a_1}) = 27/4$, $E_{\Pr}(u_{a_2}) = 31/4$, $E_{\Pr}(u_{a_3}) = 16/4$, $E_{\Pr}(u_{a_4}) = 26/4$, so $a_2 \succ a_1 \succ a_4 \succ a_3$ according to the principle of insufficient reason.

(b) If we multiply the utilities by 2 and add 5, then we are performing a positive affine transformation. All the rules we have considered are unaffected by positive affine transformations, so none change.

(c) Replacing n by 10^n is an increasing function, so maximin is unaffected. However, all the other rules may be affected. In fact, in this case, they are. For the other three rules, it's easy to see that the best answer determines the order, so we get $a_4 \succ a_2 \succ a_1 \succ a_3$ is the preferece order for opt and the principle of insufficient reason, and $a_4 \succ a_2 \sim a_1 \succ a_3$ is the order for regret. Thus, the orders are different from those in (a) for all cases other than maximin.

Grading: For undergrads, 20 points were given for part (a) (5 for each subpart) and 10 for part (b). For grads, 14 points were given for part (a) (3–4–4–3), 6 for part (b), and 10 for part (c). If you showed some work for part (a), you got significant partial credit if you made silly or arithmetic mistakes; if you didn't show work, it was hard to give partial credit! For part (b), you needed to explain that the orders didn't change because the transformation was a *positive* affine transformation. (The "positive" part is because we multiply by 2; if you multiply by a negative number, things do change, in general.) For part (c), it wasn't enough just to say that the ordering was affected in all cases but maximin, you had to say what was affected. Finally, a minor comment: $\sim$ is not the same as $=$! If you write $a \sim a'$, that means that the decision maker is indifferent between a and a'. if you write $a = a'$, you mean that these are the same acts (i.e., the same functions from states to outcomes), which is a much stronger statement than saying that the decision maker is indifferent between them.

III (40 points)

Let $X = \{(i,j) : i \text{ and } j \text{ are non-negative integers.}\}$. Suppose that $x \succ y$ if and only if $x_1 + x_2 > y_1 + y_2 + 2$.

(a) Show that $\succ$ is transitive.

Solution: if $(i_1, j_1) \succ (i_2, j_2)$ and $(i_2, j_2) \succ (i_3, j_3)$, then $i_1 + j_1 > i_2 + j_2 + 2$ and $i_2 + j_2 > i_3 + j_3 + 2$. Hence $i_1 + j_1 - 2 > i_2 + j_2 > i_3 + j_3 + 2$, so $(1_1, j_1) \succ (i_3, j_3)$, as desired.

(b) Find a weak representation for $\succ$, i.e. a function $u : X \rightarrow R$ such that if $x \succ y$ then $u(x) > u(y)$.

Solution: This is easy: just take $u(i,j) = i + j$. Clearly if $(i_1, j_1) > (i_2, j_2)$, then $i_1 + j_1 > i_2 + j + 2 + 2$, so we certainly have $i_1 + j_1 > i_2 + j_2$.

(c) Show that there does not exist a utility representation for $\succ$.

Solution: As usual, define $x \succeq y$ as $\neg(y \succ x)$ and define $x \sim y$ as $x \succeq y$ and $y \succeq x$. If there were a representation u, then we would have $x \succeq y$ iff $u(x) \succeq u(y)$ and $x \sim y$ iff $u(x) = u(y)$. Since $=$ is transitive, it would have to be the case that $\sim$ is transitive, but it's not. For example, it's easy to see that $(1,1) \sim (1,3)$ and $(1,3) \sim (1,5)$, but it's not the case that $(1,1) \sim (1,5)$. Thus, there can't be a representation.

(d) Let $A = \{x \in X : x_1 + 2x_2 \leq 10\}$. What is $C(A)$?

Solution: Note that

$$\begin{aligned} A = \; & \{(10,0),(9,0),\ldots,(0,0),(8,1),(7,1),\ldots,(0,1),(6,2),(5,2),\ldots, \\ & (0,2),(4,3),\ldots,(0,3),(2,4),(1,4),(0,4),(0,5)\}. \end{aligned}$$

The undominated elements are (10,0), (9,0), (8,0), (8,1), (7,1), and (6,2) (for example, (10,0) does not dominate (6,2) because it is not the case that $10 + 0 > 6 + 2 + 2$). Thus, $C(A) = \{(10,0),(9,0),(8,0),(8,1),(7,1),(6,2)\}$.