1. Exercises on null sets:
   (a) Show that $\emptyset$ is null.
   (b) Show that if $B \subseteq A$ and $A$ is null, then $B$ is null.
   (c) Show that if $A$ and $B$ are disjoint and null, then $A \cup B$ is null.

2. GRAD: Let $S$ denote a set of states, $O$ a set of outcomes, $L$ the set of all Savage acts from $S$ to $O$, and $\succ$ a preference order on $L$. Suppose that $\succ$ has an expected utility representation with payoff function $u$ and probability distribution $p$. Show that if $A$ is not a null event, then $f \succ_A g$ iff the conditional expected utility of $f$ given $A$ exceeds that of $g$ given $A$.

3. Show exactly where the standard choices made fail the Independence Postulate in (a) the Allais Paradox and (b) Ellsberg’s paradox. (Note that different versions of the Independence Postulate are involved; the first is one is for von Neumann-Morgenstern, the second is for Savage.)

4. Let $A_4^w$ be the following weakening of $A_4$:

   $A_4^w$. If $p$ and $q$ are probabilities on prizes and $s$ and $s'$ are non-null, then $p_{\{s\}} f \succ q_{\{s\}} f$ implies $p_{\{s'\}} f \succeq q_{\{s'\}} f$.

   The key difference between $A_4$ and $A_4^w$ is that in $A_4^w$, the conclusion has “$\succeq$”, not $\succ$.

   Show that MMEU satisfies $A_2$ and $A_4^w$. More precisely, given a set $P$ of probability measure and a utility function $u$, consider the preference order $\succ_P^u$ induced by MMEU: $f \succ_P^u g$ iff $\inf_{P \in P} E_P(u \circ f) > \inf_{P \in P} E_P(u \circ g)$ Show that $\succ_P^u$ satisfies $A_2$ and $A_4^w$. For extra credit, show that by means of a counterexample that MMEU does not in general satisfy $A_4$ (i.e., $A_4^w$ with $\succeq$ replaced by $\succ$).
5. Prove that, for a probability measure $\mu$ and a random variable mapping a finite state space $S$ to the real numbers, the definition of expectation given by Choquet coincides with the standard definition; that is, if the values of the function $f$ are $x_1 < x_2 < \ldots < x_n$ and $\mu$ is a probability on $S$, then

$$\sum_{s \in S} f(s)\mu(s) = x_1 + (x_2 - x_1)\mu(f > x_1) + \cdots + (x_n - x_{n-1})\mu(f > x_{n-1}).$$

(As usual, $f > x_i$ is the set $\{s \in S : f(s) > x_i\}$.)

6. **GRAD:** This exercise shows that A1 and A4 imply A4'. So assume that $\succ$ is an order on acts (horse lotteries) for which A1 and A4 hold. If $p$ is a lottery over prizes, let $\overline{p}$ denote the constant function on states that always gives $p$. Recall that $f_Xg$ is the act that agrees with $f$ on $X$ and with $g$ on $X^c$.

(a) Show that if $\overline{p}(s)^{f} \succ \overline{q}(s)^{f}$ and $s$ is non-null, then $\overline{p} \succ \overline{q}$. (Hint: Start by showing that if $\overline{p}(s)^{f} \succ \overline{q}(s)^{f}$ and $s$ and $s'$ are non-null, then $\overline{p}(s')^{f'} \succ \overline{q}(s')^{f'}$ for an arbitrary act $f'$. The case that $f = f'$ is just A4. Proceed by induction on $|D(f, f')|$, where $D(f, f')$ is the number of states where $f$ and $f'$ differ. Then show that for all sets $S'$ such that $S'$ is non-null, $\overline{p}_{S'}^{f} \succ \overline{q}_{S'}^{f'}$. This can be done easily by induction on $S'$. Finally, observe that $\overline{p}_S f = \overline{p}$ and $\overline{q}_S f = \overline{q}$.)

(b) Prove A4': show that if $\overline{f}(s) \succeq \overline{g}(s)$ for all $s \in S$, then $f \succeq g$. (Hint: Show by induction on $|S'|$ that if $\overline{f}(s) \succeq \overline{g}(s)$ for all $s \in S$, then $f_{S'} g \succeq g$.)