Problem Set 3:

1. Suppose that two gambles pay off one of three prizes: 30, 50, or 70 dollars. Gamble I pays off with probabilities 1/4, 1/4, and 1/2 on the three prizes, respectively, and gamble II pays off with probabilities 1, 0, and 0; that is, it pays out 30 dollars for sure. A gambler is an expected utility maximizer with payoff function \( u(x) = x - bx^2 \).

(a) Which gamble is preferred when \( b = 0.005 \)?
(b) Which is preferred when \( b = 0.01 \)?
(c) For what value of \( b \) would the gambler be indifferent between gambles I and II?
(d) If you saw the gambler choose gamble II, what could you infer about the value of his \( b \)?

2. GRAD: Suppose that an individual has a utility function of wealth, \( u(w) = -e^{-\lambda w} \). Suppose that \( \tilde{w} \) (tilde \( w \) for those of you reading on a screen) is normally distributed with mean \( \mu_w \) and variance \( \sigma^2_w \). Prove that the expected utility of \( \tilde{w} \) is \( E\{u(\tilde{w})\} = -e^{-\lambda(\mu_w - \lambda \sigma^2_w/2)} \).

3. Suppose that an individual has a total wealth of \( w_0 \). He can invest money in two assets. One, money, costs 1 dollar per unit and pays out 1 dollar per unit, for sure. The other, risky asset, costs, \( p \) dollars per unit, and pays out \( \tilde{r} \) dollars per unit (tilde \( r \) for those of you reading on a screen). The parameter \( \tilde{r} \) is a random variable that is normally distributed with mean \( \bar{r} \) (\( r \) bar for those on a screen) and variance \( \sigma^2_r \).

(a) Suppose that the individual holds a portfolio containing \( m \) units of money and \( x \) units of the risky asset. What is the mean and variance of the payout of his portfolio?
(b) Suppose that he is constrained in choosing a portfolio by a budget constraint: The cost of his portfolio must equal his initial wealth \( w_0 \). Suppose that he chooses a risky asset holding of \( x \) units. What is the mean and variance of his portfolio, expressed in terms of \( w_0 \), \( p \), \( x \), and the mean and variance of \( \bar{r} \)?
(c) Suppose that his utility function of wealth is that of problem 1. What is the expected utility of buying \( x \) units of the risky asset?
(d) Compute the optimal choice of the risky asset, the expected-utility-maximizing portfolio. Do not worry about the fact that $x$ could be negative. We allow “short sales”.

(e) Compute for this utility function the coefficient of absolute risk aversion, $-u''(w)/u'(w)$. How does the optimal portfolio change as the coefficient of absolute risk aversion increases?

4. Suppose that an expected utility maximizer faces a decision problem in which there are two states of nature and three choices, $a_1, \ldots, a_3$. Utility payoffs are described in the following table: The true probability distribution is $p = (p_1, p_2)$, where $p_s$ is the probability of state $s$.

<table>
<thead>
<tr>
<th></th>
<th>$s_1$</th>
<th>$s_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_1$</td>
<td>0</td>
<td>-8</td>
</tr>
<tr>
<td>$a_2$</td>
<td>-10</td>
<td>0</td>
</tr>
<tr>
<td>$a_3$</td>
<td>-4</td>
<td>-3</td>
</tr>
</tbody>
</table>

(a) The DM does not know $p$, and believes that it is equally likely that $p_1 = 1/4$ and $p_1 = 3/4$. Given these a priori beliefs about the models, what probability does she assign to the event $s_1$?

(b) Which $a_i$ will she choose?

(c) Before she chooses, she is told that the previous draw from the current distribution was $s_1$. Draws are independent, and her a priori belief, as before, is that the models are equally likely. What will she choose?

(d) Suppose instead that she is told that $s_2$ was drawn. What will she choose?

(e) How much is it worth to her, in utility terms, to know the value of the last draw (given that her prior beliefs are that both modes are equally likely)? (Hint: In part (c) you computed her expected utility if she is told $s_2$. In part (b) you computed her expected utility if she is told $s_1$. Before you are told anything, you have beliefs about how likely you are to be told $s_1$ and $s_2$. So you can compute your expected expected utility [this is not a typo; it really is “expected expected utility”] before you are told anything. From this, you can compute the value of information: the value
of knowing the value of the last draw. This notion of value of information is widely used.)

5. There is a deck with three cards:

- one card is black on both sides,
- one card is white on both sides, and
- one is black on one side and white on the other.

Alice chooses a card from the deck and puts it on the table with a black side showing.

(a) What is the probability, according to Bob, that the other side is black? Give at least two answers to the problem, and describe the protocol that generates them.

(b) What if Bob doesn’t know Alice’s protocol (which is probably the case in practice). What would be a good way to model the problem in that case?

6. You’re trying to decide whether or not to spend the morning studying for an afternoon test. You don’t particularly like studying, but you definitely want to do well on the test. Suppose for simplicity that you get utility 5 if you don’t study and do well, 4 if you study and do well, 0 if you don’t study and don’t do well, and 1 if you study and don’t do well. You have previous experience showing that studying is highly correlated with doing well on tests: the probability of doing well given that you study is 0.8, and the probability that you do well if you don’t study is 0.1. On the other hand, you didn’t get much sleep last night, and you know that typically when you don’t sleep well, you neither study (you’re too tired) nor do you do well on tests.

(a) Describe two causal scenarios: one in which not studying causes poor performance on tests and one in which lack of sleep causes both not studying and poor performance. In both scenarios, define causal probabilities that result in the correlation between studying and doing well given above.

(b) Given the probabilities used in part (a), what is the expected utility of studying in each causal model.

(c) What information would you need to distinguish the two models?
7. Consider the Bayesian network in Figure 1(a), which involves three Boolean random variables (i.e., the random variables have two truth values, true and false):

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    A
   /\ 
  /   \
B    C
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Figure 1: The Bayesian networks for Problem 7

Suppose that the Bayesian network has the following conditional probability tables (where $X$ and $\overline{X}$ are abbreviations for $X = \text{true}$ and $X = \text{false}$, respectively):

$\Pr\{A\} = 0.1$
$\Pr\{B|A\} = 0.8 \quad \Pr\{B|\overline{A}\} = 0.2$
$\Pr\{C|A\} = 0.4 \quad \Pr\{C|\overline{A}\} = 0.7$

(a) What is $\Pr\{\overline{B} \cap C|A\}$?
(b) What is $\Pr\{A|B \cap \overline{C}\}$?
(c) Suppose that we add a fourth node labeled $D$ to the network, with edges from both $B$ and $C$ to $D$, as shown in Figure 1(b). For the new network,
   i. What conditions have to hold for the Bayesian network to qualitatively represent $\Pr$?
   ii. Is $A$ conditionally independent of $D$ given $C$?
   iii. Is $C$ conditionally independent of $B$ given $A$?
In cases (ii) and (iii), explain your answer.