Objective Expected Utility

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Decision Theory I

Choice under Uncertainty

Z A finite set of outcomes.

P The set of probabilities of Z. $p \in P$ is a vector $p = (p_1, \ldots, p_n)$, $p_i \geq 0$ and $\sum_{i=1}^n p_i = 1$.

$\succ$ A binary relation on P.

Definition: An expected utility representation of $\succ$ is a payoff function $u : Z \to \mathbf{R}$ such that for $p, q \in P$, $p \succ q$ if and only if

$$\sum_{z \in Z} p(z)u(z) > \sum_{z \in Z} q(z)u(z).$$

Choice under Uncertainty

Definition: For $0 < \alpha < 1$, the α mixture of p and q in P is the probability distribution $\alpha p + (1-\alpha)q$.

Axiom 1: $\succ$ is a preference relation.

Axiom 2: For all $p, q, r \in P$, if $p \succ q \succ r$, then there exist $\alpha, \beta \in (0,1)$ such that $\alpha p + (1-\alpha)r \succ q \succ \beta p_(1-\beta)r$.

A1 and A2 imply the existence of a utility representation for $\succ$.

A2 is called an Archimedean axiom. Suppose p is 0 for sure, q is 1 penny for sure and r is death. Perhaps for all $\alpha \in (0,1)$, $p \succ \alpha q + (1-\alpha)r$, but $q \succ p$. This violates A2.

Choice under Uncertainty

Axiom 3: For $p, q, r \in P$ and $\alpha \in (0, 1]$, if $p \succ q$ then $\alpha p + (1 - \alpha)r \succ \alpha q + (1 - \alpha)r$.

A3 is called the independence axiom. This axiom suggests that indifference curves should be linear.

Choice under Uncertainty

Theorem: A binary relation $\succ$ on P has an expected utility representation iff it satisfies axioms A1 through A3. Payoff functions u and v are both expected utility representations for $\succ$ iff there are scalars $a > 0$ and b such that $v(z) = au(z) + b$.

Choice Under Uncertainty

1. There are best and worst choices $b, w \in P$. If they are not ranked, then take $u(z) \equiv 0$. Otherwise, $b \succ w$.
2. For $1 > \alpha > \beta > 0$, $\alpha b + (1-\alpha)w \succ \beta b + (1-\beta)w$.
3. For any $q \in P$ there is a unique solution α_q to the "equation" $\alpha_p + (1-\alpha)w \sim q$. $p \succ q$ iff $\alpha_p > \alpha_q$.
4. Define $v(p) = \alpha_p$. According to (3), this represents $\succ$.
5. $v(p)$ is affine: $v\big(\alpha p + (1-\alpha)q\big) = \alpha v(p) + (1-\alpha)v(q)$.
6. Let δ_z denote the lottery that pays off z for sure. Define $u(z) = v(\delta_z)$.

Choice Under Uncertainty

Take $Z = \mathbf{R}$. Let p be a probability on Z, E_p the expected value of p.

Definition: A decision maker is risk averse iff for all $p \in P$, $\delta_{E_p} \succeq p$.

Theorem: An expected utility-maximizing decisin maker is risk averse iff the payoff function is concave.

The degree of concavity reflects how much the decision maker dislikes risk.

Definition: The coefficient of absolute risk aversion of a payoff function u is $\lambda(z) \equiv = -u''(z)/u'(z)$.

Choice Under Uncertainty

Portfolio Choice

- One risk-free asset (money) m, with a total return of 1.
- One risky asset (stock) x, with a per-unit return $\tilde{r}$ which is normally distributed with mean r and variance σ^2.
- The investor has vNM preferences is $u(z) = -\exp\{-\lambda z\}$.
- The coefficient of absolute risk aversion is $\lambda > 0$.
- If $\tilde{z}$ is normally distributed with mean μ and variance σ^2, $Eu(\tilde{z}) = -\exp -\lambda(\mu - \lambda\sigma^2/2)$.

Choice Under Uncertainty

Suppose individuals have initial wealth w_0, and suppose p is the price of a unit of risky asset. The combinations of money m and asset x the investor can buy solves $w_0 = m + px$. The return on the portfolio (m, x) is $\tilde{z} = m + \tilde{r}x$. Thus the investor solves the following optimization problem:

$$\max_x E\left\{-\exp\{-\lambda(w_0 - px + \tilde{r}x)\}\right\}$$
$$\max_x -\exp\{-\lambda(w_0 - px + rx - \lambda x^2\sigma^2/2)\}$$
$$\max_x w_0 - (r - p)x - \lambda x^2\sigma^2/2$$

The maximand is concave — the first-order condition is sufficient:

$$r - p - \lambda x\sigma^2 = 0$$
$$x^* = \frac{r - p}{\lambda\sigma^2}$$