



Constructive Decision Theory

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Savage's Approach

Savage's approach to decision making has dominated decision theory since the 1950's. It assumes that a decision maker (DM) is given/has

- a set S of states
- a set O of outcomes

A (Savage) *act* is a function from states to outcomes.

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Example: Betting on a horse race.

- S = possible orders of finish
- O = how much you win
- act = bet

Savage's Theorem

Savage assumes that a DM has a preference order \succsim on acts satisfying certain postulates:

- E.g. transitivity: if $a_1 \succsim a_2$ and $a_2 \succsim a_3$, then $a_1 \succsim a_3$.

He proves that if a DM's preference order satisfies these postulates, then the DM is acting as if

- he has a probability P_r on states
- he has a utility function u on outcomes
- he is maximizing expected utility:
 - $a \succsim b$ iff $E_{P_r}[u_a] \geq E_{P_r}[u_b]$.
 - $u_a(s) = u(a(s))$: the utility of act a in state s

Are Savage Acts Reasonable?

Many problems have been pointed out with Savage's framework. We focus on one:

- People don't think of acts as function from states to outcomes
- In a complex environment, it's hard to specify the state space and outcome space before even contemplating the acts
 - What are the states/outcomes if we're trying to decide whether to attack Iraq?
- What are the acts if we can't specify the state/outcome space?

Acts as Programs

An alternative: instead of taking acts to be functions from states to outcomes, we take acts to be syntactic objects

- essentially, acts are *programs* that the DM can run.

Consider the act “Buy 100 shares of IBM”:

- Call the stock broker, place the order, . . .

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Program can also have *tests*

- **if** the Democrats win **then** buy 100 shares of IBM

To specify tests, we need a *language*

The Setting

Savage assumes that a DM is given a state space and an outcome space. We assume that the DM has

- a set \mathcal{A}_0 of primitive programs
 - Buy 100 shares of IBM
 - Attack Iraq
- a set T_0 of primitive tests (i.e., formulas)
 - The price/earnings ratio is at least 7
 - The moon is in the seventh house
- a theory AX
 - Some axioms that describe relations between tests
 - E.g., $t_1 \Leftrightarrow t_2 \wedge t_3$

The Programming Language

In this talk, we consider only one programming construct:

- **if ... then ... else**

- If a_1 and a_2 are programs, and t is a test, then

- **if t then a_1 else a_2** is a program

- **if moon in seventh house then buy 100 shares IBM**

- tests formed by closing off T_0 under conjunction and negation:

- tests are just propositional formulas

Let \mathcal{A} denote this set of programs (acts).

In the full paper we also consider randomization.

- With probability r perform a_1 ; with probability $1 - r$, perform a_2

Programming Language Semantics

What should a program *mean*?

In this paper, we consider *input-output* semantics:

- A program defines a function from states to outcomes
 - once we are given a state space and an outcome space, a program determines a Savage act
- The state and outcome spaces are now subjective.
 - Different agents can model them differently
- The agent's theory AX affects the semantics:
 - interpretation of tests must respect the axioms

Semantics: Formal Details I

Given a state space S and an outcome space O , we want to view a program as a function from S to O , that respects AX. We first need

- a *program interpretation* ρ_{SO} that associates with each primitive program in \mathcal{A}_0 a function from S to O
- a *test interpretation* π_S that associates with each primitive proposition in T_0 an event (a subset of S)
 - extend to T in the obvious way
 - require that $\pi_S(t) = S$ for each axiom $t \in \text{AX}$
 - axioms are necessarily true

Can extend ρ_{SO} to a function that associates with each program in \mathcal{A} a function from S to O :

$$\rho_{SO}(\mathbf{if } t \mathbf{ then } a_1 \mathbf{ else } a_2)(s) = \begin{cases} \rho_{SO}(a_1)(s) & \text{if } s \in \pi_S(t) \\ \rho_{SO}(a_2)(s) & \text{if } s \notin \pi_S(t) \end{cases}$$

Where We're Headed

We prove the following type of theorem:

If a DM has a preference order on programs satisfying appropriate postulates, then there exist

- a state space S ,
- a probability Pr on S ,
- an outcome space O ,
- a utility function u on O ,
- a program interpretation ρ_{SO} ,
- a test interpretation π_S

such that $a \succeq b$ iff $E_{\text{Pr}}[u_{\rho_{SO}(a)}] \geq E_{\text{Pr}}[u_{\rho_{SO}(b)}]$.

● This is a Savage-like result

- The postulates are variants of standard postulates
- The DM has to put a preference order only on “reasonable” acts

But now S and O are subjective, just like Pr and u !

- S , O , Pr , u , ρ_{SO} , and π_S are all in the DM's head
- S and O are not part of the description of the problem

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We have replaced Savage acts by programs and prove Savage-type theorems. So what have we gained?

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We have replaced Savage acts by programs and prove Savage-type theorems. So what have we gained?

- Acts are easier for a DM to contemplate
 - No need to construct a state space/outcome space
 - Just think about what you can do
- Different agents can have different conceptions of the world
 - You might make decision on stock trading based on price/earnings ratio
 - I might use astrology (and might not even understand the notion of p/e ratio)

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- Can deal with unanticipated events, novel concepts:
 - Updating \neq conditioning
- We do not have to identify two acts that act the same as functions
 - Can capture resource-bounded reasoning (agent can’t tell two acts are equivalent)
 - allow nonstandard truth assignments
 - $t_1 \wedge t_2$ may not be equivalent to $t_2 \wedge t_1$
- Can capture framing effects

Framing Effects

Example: [McNeill et al.] DMs are asked to choose between surgery or radiation therapy as a treatment for lung cancer. They are told that,

- Version 1: of 100 people having surgery, 90 alive after operation, 68 alive after 1 year, 34 alive after 5 years; with radiation, all live through the treatment, 77 alive after 1 year, 22 alive after 5 years
- Version 2: with surgery, 10 die after operation, 32 dead after one year, 66 dead after 5 years; with radiation, all live through the treatment, 23 dead after one year, 78 dead after 5 years.

Both versions equivalent, but

- In Version 1, 18% of DMs prefer radiation;
- in Version 2, 44% do

Framing in our Framework

Primitive propositions:

- RT : 100 people have radiation therapy;
- S : 100 people have surgery;
- $L_0(k)$: $k/100$ people live through operation ($i = 0$)
- $L_1(k)$: $k/100$ are alive after one year
- $L_5(k)$: $k/100$ are alive after five years
- $D_0(k)$, $D_1(k)$, $D_5(k)$ similar, with death

Primitive programs

- a_S : perform surgery (primitive program)
- a_R : perform radiation therapy

- Version 1: Which program does the DM prefer:

$$a_1 = \text{if } t_1 \text{ then } a_S \text{ else } a, \text{ or}$$

$$a_2 = \text{if } t_1 \text{ then } a_R \text{ else } a,$$

where a is an arbitrary program and

$$t_1 = (S \Rightarrow L_0(90) \wedge L_1(68) \wedge L_5(34)) \wedge$$

$$(RT \Rightarrow L_0(100) \wedge L_1(77) \wedge L_5(22))$$

- Can similarly capture Version 2, with analogous test t_2 and programs b_1 and b_2
- Perfectly consistent to have $a_1 \succ a_2$ and $b_2 \succ b_1$
- A DM does not have to identify t_1 and t_2
 - Preferences should change once $t_1 \Leftrightarrow t_2$ is added to theory

The Cancellation Postulate

Back to the Savage framework:

Cancellation Postulate: Given two sequences $\langle a_1, \dots, a_n \rangle$ and $\langle b_1, \dots, b_n \rangle$ of acts, suppose that for each state $s \in S$

$$\{\{a_1(s), \dots, a_n(s)\}\} = \{\{b_1(s), \dots, b_n(s)\}\}.$$

• $\{\{o, o, o, o', o'\}\}$ is a *multiset*

If $a_i \succeq b_i$ for $i = 1, \dots, n - 1$, then $b_n \succeq a_n$.

Cancellation is surprising powerful. It implies

- Reflexivity

- Transitivity:

- Suppose $a \succeq b$ and $b \succeq c$. Take $\langle a_1, a_2, a_3 \rangle = \langle a, b, c \rangle$ and $\langle b_1, b_2, b_3 \rangle = \langle b, c, a \rangle$.

- Event independence:

- Suppose that $T \subseteq S$ and $f_T g \succeq f'_T g$

- $f_T g$ is the act that agrees with f on T and g on $S - T$.

- Take $\langle a_1, a_2 \rangle = \langle f_T g, f'_T h \rangle$ and $\langle b_1, b_2 \rangle = \langle f'_T g, f_T h \rangle$.

- Conclusion: $f_T h \succeq f'_T h$

Cancellation in Our Framework

A program maps truth assignments to primitive programs:

● E.g., consider **if t then a_1 else (if t' then a_2 else a_3)**:

● $t \wedge t' \rightarrow a_1$

● $t \wedge \neg t' \rightarrow a_1$

● $\neg t \wedge t' \rightarrow a_2$

● $\neg t \wedge \neg t' \rightarrow a_3$

Similarly for every program.

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Can rewrite the cancellation postulate using programs:

● replace “outcomes” by “primitive programs”

● replace “states” by “truth assignments”

The Main Result

Theorem: Given a preference orders \succeq on acts satisfying Cancellation, there exist

- a set S of states and a set \mathcal{P} of probability measures on S ,
- a set O of outcomes and a utility function u on O ,
- a program interpretation ρ_{SO} ,
- a test interpretation π_S

such that

$$a \succeq b \text{ iff } E_{\text{Pr}}[u_a] \geq E_{\text{Pr}}[u_b] \text{ for all } \text{Pr} \in \mathcal{P}.$$

Moreover, if \succeq is totally ordered, then \mathcal{P} can be taken to be a singleton.

Updating

In the representation, can always take the state space to have the form

$AT \times TOT(\succeq)$:

- AT = all truth assignments to tests
- $TOT(\succeq)$ = total orders extending \succeq

Updating proceeds by conditioning:

- Learn $t \Rightarrow$ representation is $\mathcal{P} \mid t$
- Learn $a \succeq b$: representation is $\mathcal{P} \mid (\succeq \oplus (a, b))$

Uniqueness

Savage gets uniqueness; we don't:

- We do have a canonical representation $AT \times TOT(\succeq)$
- In the totally ordered case, $S = AT$.
- Cannot take $S = AT$ in the partially-ordered case
 - Even with no primitive propositions, if primitive programs a and b are incomparable, need two states, two outcomes, and two probability measures to represent this.
- Can't hope to have a unique probability measure on S , even in the totally ordered case: there aren't enough acts.
 - Savage's postulates force uncountably many acts

Program Equivalence

When are two programs *equivalent*?

- That depends on the choice of semantics
- With input-output semantics, two programs are equivalent if they determine the same functions *no matter what* S , O , π_S , ρ_{SO} are.

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Example 1: $(\text{if } t \text{ then } a \text{ else } b) \equiv (\text{if } \neg t \text{ then } b \text{ else } a)$.

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Example 2: If $t \equiv t'$, then

$(\text{if } t \text{ then } a \text{ else } b) \equiv (\text{if } t' \text{ then } a \text{ else } b)$.

Cancellation and Equivalence

Testing equivalence of propositional formulas is hard

- co-NP complete, even for this simple programming language
- Have to check propositional equivalence

Cancellation implies a DM is indifferent between equivalent programs.

Lemma: Cancellation \Rightarrow if $a \equiv b$, then $a \sim b$.

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Lemma: Cancellation \Rightarrow if $a \equiv b$, then $a \sim b$.

- Cancellation requires smart decision makers!
- We don't have to require cancellation
 - Can consider more resource-bounded DM's

Conclusions

The theorems we have proved show only that this approach generalizes the classic Savage approach.

- The really interesting steps are now to use the approach to deal with issues that the classical approach can't deal with
 - conditioning on unanticipated events
 - (un)awareness
 - papers with Rêgo
 - learning concepts
 - ...