Static Decision Theory Under Certainty

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- A set of objects X
- An individual is asked to express preferences among the objects, or to make choices from subsets of X.
- For $x, y \in X$ we can ask which, if either, is strictly preferred, that is, the best of the two.
- If the subject says, "I prefer x to y," then we write $x \succ y$ and say, "x is strictly preferred to y."
- The relation $\succ$ is a binary relation.

Example 1: $X = \{a, b, c\}$, $b \succ a$, $a \succ c$, and $b \succ c$. What if the subject also says $a \succ b$?

Axioms – properties that (arguably) all preference orders should satisfy.

Asymmetry: For all $x, y \in X$, if $x \succ y$ then $y \not\succ x$.

Negative Transitivity: For all $x, y, z \in X$, if $x \not\succ y$ and $y \not\succ z$ then $x \not\succ z$.

Proposition: The binary relation $\succ$ is negatively transitive iff $x \succ z$ implies that for all y, $y \succ z$ or $x \succ y$.

Example 2: $X = \{a, b, c\}$, $b \succ a$, $a \succ c$ and $b\, ?\, c$. Asymmetry and NT you also know how b and c must be ranked.

Definition: A binary relation $\succ$ is called a (strict) preference relation if it is asymmetric and negatively transitive.

Is asymmetry a good normative or descriptive property? What about negative transitivity.

Definition: For $x, y \in X$,

- $x \succ y$ iff $y \not\succsim x$;
- $x \sim y$ iff $y \not\succ x$ and $x \not\succ y$.

Does the absence of strict preference in either direction require real indifference or could it permit non-comparability?

Example: $X = \{a, b, c\}$. Suppose a is not ranked (by $\succ$) relative to either b or c. If $\succ$ satisfies NT, then b and c are not ranked either.

Definition: The binary relation $\succeq$ on X is complete if for all $x, y \in X$, $x \succeq y$ or $y \succeq x$. $\succeq$ is transitive iff for all $x, y, z \in X$, $x \succeq y$ and $y \succeq z$ implies $x \succeq z$.

Proposition: Let $\succ$ be a binary relation on X.

- $\succ$ is asymmetric iff $\succeq$ is complete.
- $\succ$ is negatively transitive iff $\succeq$ is transitive.

Proof: $\Longrightarrow$

- Asymmetry implies that for no pair $x, y \in X$ is it true that both $x \succ y$ and $y \succ x$. Thus at least one of $x \nsucc y$ and $y \nsucc x$ must hold. So at least one of $x \succeq y$ and $y \succeq x$ is true. That is, $\succeq$ is complete.
- If $x \nsucc y$ and $y \nsucc z$, then $x \nsucc z$. By definition we have $y \succeq x$ and $z \succeq y$ implies $z \succeq x$, so $\succeq$ is transitive.

$\Longleftarrow$ will be on homework 1.

Proposition: If $\succ$ is a preference relation, then $\succ$ is transitive.

Is transitive a useful property?

- Normative property?
- The coffee cup example.
- Without transitivity, there may be no preference maximal object in a set of alternatives.

Suppose that X is finite. Let $P^+(X)$ denote the set of all non-empty subsets of X.

Definition: A choice function is a function $c : P^+(X) \to P^+(X)$ such that for all $A \in P^+(X)$, $c(A) \subseteq A$.

$c(A)$ is the set of objects "chosen" from A.

Preference relations define choice functions.

Definition: For a preference relation $\succ$ on X, its choice function $c_\succ : P^+(X) \to P^+(X)$ is

$$c_\succ(A) = \{x \in A : \text{ for all } y \in A, y \not\succ x\}.$$

Things to think about:

- Show that if $x, y \in c_{\succ}(A)$, then $x \sim y$.
- Show that for all $A \in P^{+}(X)$, $c_{\succ}(A) \neq \emptyset$.

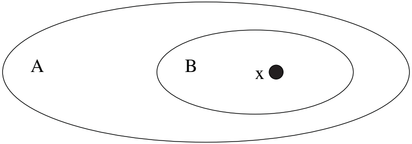
The second item justifies the use of the phrase choice function to describe $c_{\succ}$.

For every choice function c is there a preference order $\succ$ such that $c = c_{\succ}$?

Clearly not:

Example: $X = \{a, b, c\}$.

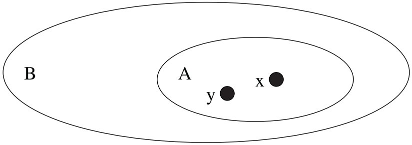
- $c\big(\{a, b, c\}\big) = \{a\}$ and $c\big(\{a, b\}\big) = \{b\}$ violates asymmetry.
- $c\big(\{a, b\}\big) = \{a\}$ and $c\big(\{b, c\}\big) = \{b\}$ and $c\big(\{a, c\} = \{c\}$ violates negative transitivity.



Axiom α: If $x \in B \subset A$ and $x \in C(A)$, then $x \in C(B)$.

Proposition: If $\succ$ is a preference relation, then $c_{\succ}$ satisfies axiom α.

Proof: Suppose there are sets $A, B \in P^{+}(X)$ satisfying the hypotheses, that $x \in c_{\succ}(A)$ and $x \notin c_{\succ}(B)$, Then there is a $y \in B$ such that $y \succ x$. Since $B \subset A$, $y \in A$ and so $x \notin c_{\succ}(A)$, contrary to our hypothesis.



Axiom β: If $x, y \in c(A)$, $A \subset B$ and $y \in c(B)$, then $x \in c(B)$.

Proposition: If $\succ$ is a preference relation, then $c_{\succ}$ satisfies axiom β.

Proof: Since $\in c_{\succ}(A)$ and $y \in A$, $y \not\succ x$. Since $y \in c_{\succ}(B)$, for all $z \in B$, $z \not\succ y$. Negative transitivity implies that for all $z \in B$, $z \not\succ x$. Thus $x \in c_{\succ}(B)$.

Axioms α and β **characterize** preference-based choice.

Proposition: If a choice function c satisfies axioms α and β, then there is a preference relation $\succ$ such that $c = c_{\succ}$.

Proof: Two steps

- Define a “revealed preference order” $\succ$ and show that it is a preference relation, i.e. asymmetric and negatively transitive.
- Show that $c = c_{\succ}$.

Define a preference order: $x \succ y$ iff $x \neq y$ and $c(\{x, y\}) = \{x\}$. Notice that, by definition, $x \not\succ x$.

- $\succ$ is asymmetric.

  Suppose not. Suppose $x \succ y$ and $y \succ x$. Then $c(\{x, y\}) = \{x\}$ and $c(\{x, y\}) = \{y\}$. But both cannot be true.

- $\succ$ is negatively transitive.

  Suppose that for some $x, y, z \in X$, $z \not\succ y$ and $y \not\succ x$. Show that $z \not\succ x$. That is, show that $x \in c(\{x, z\})$. It suffices to show $x \in c(\{x, y, z\})$, because then $x \in c(\{x, z\})$ follows from α.

  Suppose that $x \notin c(\{x, y, z\})$. Then one or both of y and z are in $c(\{x, y, z\})$ because $c(\{x, y, z\}) \neq \emptyset$. We will show that neither of them can be in.

- $y \notin c(\{x, y, z\})$.

  Suppose $y \in c(\{x, y, z\})$. Axiom α implies $y \in c(\{x, y\})$. Since $y \nsucc x$, $x \in c(\{x, y\})$. Axiom β implies $x \in c(\{x, y, z\})$.

- $z \notin c(\{x, y, z\})$.

  Suppose $z \in c(\{x, y, z\})$. Axiom α implies $z \in c(\{y, z\})$. $z \nsucc y$ implies $y \in c(\{y, z\})$. Axiom β implies $y \in c(\{x, y, z\})$.

"Revealed preferred to" $\succ$ is a preference relation. Now we have to show that for all $A \in P^+(A)$, $c(A) = c_\succ(A)$.

- Suppose $x \in c(A)$.

  α implies $x \in c(\{x, y\})$ for all $y \in A$. By definition, for all $y \in A$, $y \not\succ x$. Thus $x \in c_\succ(A)$.

- Suppose $\in c_\succ(A)$.

  Then for all $y \in A$, $y \not\succ x$, and so $x \in c(\{x, y\})$. Choose $z \in C(A)$. If $z \neq x$, axiom α implies $z \in c(\{x, z\})$, so $c(\{x, z\}) = \{x, z\}$. Axiom β now implies $x \in C(A)$.

QED

An alternative characterization of preference-based choice functions:

Weak Axiom of Revealed Preference: If $x, y \in A \cap B$ and $x \in c(A)$ and $y \in c(B)$, then $x \in c(B)$ and $y \in c(A)$.

This axiom is called Houthakker's Axiom, or WARP.

Proposition: c satisfies axioms α and β iff it satisfies WARP.

Proof: **?**

We have already dissed completeness of $\succeq$.

Definition: $\succ$ is a partial order iff it is asymmetric and transitive.

Problem: Characterize $c_\succ$ for partial orders.

Axiom α still holds, but β may feel. See homework 1.

Now we do not want to define indifference as before, since the usual definition expresses both indifference and non-comparability. One could define the pair $(\succ, \sim)$ and theorize about the pair.