Problem Set 4

Handed out: Nov. 18, 2010. Due: Dec. 2, 2010

1. Exercises on null sets:
   (a) Show that $\emptyset$ is null.
   (b) Show that if $B \subseteq A$ and A is null, then B is null.
   (c) Show that if A and B are disjoint and null, then $A \cup B$ is null.

2. **GRAD:** Let S denote a set of states, O a set of outcomes, L the set of all Savage acts from S to O, and $\succ$ a preference order on L. Suppose that $\succ$ has an expected utility representation with payoff function u and probability distribution p. Show that if A is not a null event, then $f \succ_A g$ iff the conditional expected utility of f given A exceeds that of g given A.

3. Consider a decision problem with 2 states and 3 acts. Payoffs $u(s, d)$ are given by

	d_1	d_2	d_3
s_1	0	-10	-4
s_2	-8	0	-3

   The decisionmaker is an expected utility maximizer. The true probability distribution is $p = \big(p(s_1), p(s_2)\big)$.

   (a) The DM does not know p, and believes that it is equally likely that $p(s_1) = 1/4$ and $p(s_1) = 3/4$. Which d_i will she choose?
   (b) Before she chooses, she is told that the previous draw from the current distribution was s_1. Draws are independent, and priors are as before. What will she choose?
   (c) Suppose instead that she is told that s_2 was drawn. What will she choose?
   (d) How much is it worth to her to know the value of the last draw (given that her prior is as defined in part (b)). (Hint: In part (c),

you computed her expected utility if she is told s_2. In part (b) you computed her expected utility if she is told s_1. Before you are told anything, you have beliefs about how likely you are to be told s_1 and s_2. So you can compute your expected expected utility [this is not a typo; it really is "expected expected utility"] before you are told anything. From this, you can compute the *value of information*—the value of knowing the value of the last draw. This notion of value of information is widely used.)

4. Show exactly where the standard choices made fail the Independence Postulate in (a) the Allais Paradox and (b) Ellsberg's paradox. (Note that different versions of the Independence Postulate are involved; the first is one is for von Neumann-Morgenstern, the second is for Savage.)

5. There is a deck with three cards:

   - one is black on both sides,
   - one is white on both sides, and
   - one is black on one side and white on the other.

   Alice chooses a card from the deck and puts it on the table with a black side showing.

   (a) What is the probability, according to Bob, that the other side is black? Give at least two answers to the problem, and describe the protocol that generates them.

   (b) What if Bob doesn't know Alice's protocol (which is probably the case in practice). What would be a good way to model the problem in that case?

6. You're trying to decide whether or not to spend the morning studying for an afternoon test. You don't particularly like studying, but you definitely want to do well on the test. Suppose for simplicity that you get utility 5 if you don't study and do well, 4 if you study and do well, 0 if you don't study and don't do well, and -1 if you study and don't do well. You have previous experience showing that studying is highly correlated with doing well on tests: the probability of doing well given that you study is .8, and the probability that you do well if you don't study is .1. On the other hand, you didn't get much sleep last night,

and you know that typically when you don't sleep well, you neither study (you're too tired) nor do you do well on tests.

(a) Describe two causal scenarios: one in which not studying causes poor performance on tests and one in which lack of sleep causes both not studying and poor performance. In both scenarios, define causal probabilities that result in the correlation between studying and doing well given above.

(b) Given the probabilities used in part (a), what is the expected utility of studying in each causal model.

(c) What information would you need to distinguish the two models?

7. Consider the following Bayesian network containing 3 Boolean random variables (that is, the random variables have two truth values—*true* and *false*):

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   A
  / \
 B   C
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Suppose the Bayesian network has the following conditional probability tables (where X and $\overline{X}$ are abbreviations for $X = \mathit{true}$ and $X = \mathit{false}$, respectively):

$$\begin{aligned}
&\Pr(A) = .1\\
&\Pr(B \mid A) = .8\\
&\Pr(B \mid \overline{A}) = .2\\
&\Pr(C \mid A) = .4\\
&\Pr(C \mid \overline{A}) = .7
\end{aligned}$$

(a) What is $\Pr(\overline{B} \cap C \mid A)$?

(b) What is $\Pr(A \mid B \cap \overline{C})$?

(c) Suppose we add a fourth node labeled D to the network, with edges from both B and C to D. For the new network

(i) What conditions have to hold for the Bayesian network to qualitatively represent Pr.

(ii) Is A conditionally independent of D given C?

(iii) Is C conditionally independent of B given A?

In cases (ii) and (iii), explain your answer.