CS5625: Light Reflection

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Spring 2020

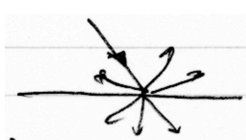
These notes are about how we describe optical reflection at surfaces, radiometrically.

We've discussed how light travels through empty space. Today we look at how it interacts with stuff in the scene—after all, if the light just travels unimpeded the pictures will be pretty uninteresting.

1 Scattering

Scattering is when light coming from one direction ends up going into a range of directions. Specular interactions with smooth metal and glass surfaces is not scattering; it's reflection or refraction. I use the term "ideal specular" when I want to be clear I'm talking about this kind of one-direction-in, one-direction-out phenomenon.

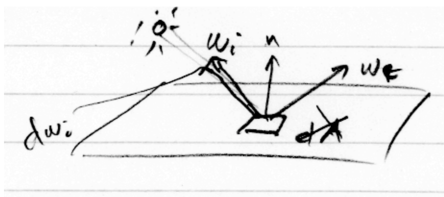
There are three kinds of descriptions of scattering that are widely used in graphics. BSDF, Bidirectional Scattering Distribution Function: describes scattering at infinitesimally thin surfaces (either the surface of a thick object or a thin object like a window or a sheet of paper). BSDF describes scattering both back to the same side (reflection, BRDF) and through to the other side (transmission, BTDF). This is a function of a point (on a surface) and two directions (incoming and outgoing, or incident and exitant).



In these notes I'll mostly be concerned with giving precise definitions of these quantities.

2 BRDF

Think of a surface, with light incident on it at a particular point. Let's consider just the light arriving from an infinitesimal solid angle $d\omega_i$ around the incoming direction ω_i. If the radiance in this solid angle is L_i, then the (infinitesimal) irradiance on the surface is $dE_i(\omega_i) = L_i\mu(d\omega_i)$. (Remember that the projected solid angle measure μ has a factor of $\mathbf{n} \cdot \omega_i$ built into it, so that it assigns a smaller measure to more grazing solid angles.)



If this is a scattering surface (one that sends light to a range of directions, not just a single direction as a mirror would), this produces a distribution of infinitesimal reflected radiance $dL_r(\omega_r)$ over the hemisphere of outgoing directions ω_r.

The BRDF is the ratio of the reflected radiance to the incident irradiance:

$$f_s(\omega_i, \omega_r) = \frac{dL_r(\omega_r)}{dE_i(\omega_i)}$$

Now, if we want to get the (finite) radiance resulting from illumination from some distribution $L_i(\omega_i)$ we can just add up the contributions of illumination from differential solid angles all over the hemisphere—that is, we integrate:

$$L_r(\omega_r) = \int_{H^2} f_r(\omega_i, \omega_r)L_i(\omega_i)d\mu(\omega_i)$$

Operational version. If you like a concrete operational definition of this as a derivative, think of the experiment of illuminating the surface with a small area light source of radiance L_i (maybe a frosted light bulb with a dimmer to control L_i and an adjustable iris in front of it to control the solid angle Ω_i it illuminates) and measuring the reflected radiance L_r with a camera. The reflected radiance will be directly proportional to both the radiance of the source and, in the limit for small solid angles, the size of the solid angle. The BRDF is just the constant of proportionality between L_r and $L_i\mu(\Omega_i)$—it is the derivative of L_r with respect to E_i.

Mathematical version. If you like a more mathematical definition, you can think of light reflection as an operation on light distributions. You hand an incident radiance distribution $L_i : H^2 \to I\!R$ to the BRDF and it hands back a reflected distribution $L_r : H^2 \to I\!R$. Because of the superposition principle, this is a *linear* operator $\mathcal{R}$ on functions over the hemisphere:

Under reasonable conditions on the operator and the functions, this type of operator can always be expressed as an integral: the output is an integral of the input multiplied by a *kernel* function.

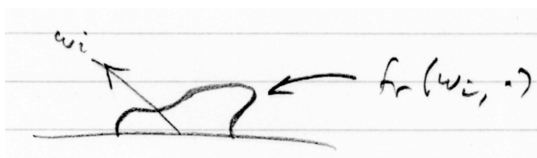
$$\mathcal{R} : (H^2 \to I\!R) \to (H^2 \to I\!R)$$
$$: L_i \mapsto \int_{H^2} f_r(\omega_i, \cdot) L_i(\omega_i) d\mu(\omega_i)$$

We call this kernel function the BRDF.

2.1 BRDFs as densities

The BRDF is a function of four variables, which makes it a bit hard to think about sometimes. If we think of it in terms of one argument at a time, that can help.

For light arriving from ω_i, the BRDF $f_r(\omega_i, \cdot)$ is the density of reflectance over the outgoing hemisphere. Reflectance is a ratio of that tells the fraction of total irradiance reflected, and the BRDF describes the *distribution* of this reflectance over the hemisphere by giving the *density* function.



If we fix the outgoing direction instead, the function $f_r(\cdot, \omega_r)$ is another kind of density. It describes the distribution of something we might call "sensitivity" over the incoming hemisphere. By this I mean: the radiance in direction ω_r depends on light coming from many directions, and $f_r(\omega_i, \omega_r)$ tells you how sensitive it is to light from the particular direction ω_i.

2.2 Units of BRDF

One question that always comes up is, What does that mean, that the BRDF has units of inverse steradians? Why can't it just be unitless, since it relates radiance out to radiance in? I have three answers to this question.

An answer that also helps explain the name: I observed above that the BRDF is a density function that measures the density of reflectance (dimensionless ratio) over the hemisphere (measured in terms of solid angle). This is a density just like population density (people per square kilometer) or mass density in a solid (grams per cubic centimeter), so it has units of "reflectance per unit solid angle." But since reflectance is dimensionless we state this unit as just "per unit solid angle" or "one over steradians" (1/sr) or "inverse steradians" (sr^{-1}).

2.3 Properties of the BRDF

Not every function of two directions makes for a good BRDF. There are two properties all BRDFs have, physically: reflection conserves energy, and they obey reciprocity.

Energy conservation The basic requirement of energy conservation is that when a surface is illuminated with total irradiance E_i then the reflected radiant exitance M_r is less than E_i. This has to be true for all distributions of irradiance, so it has to be true of irradiance coming from a small solid angle Ω_i in the direction ω_i. If we integrate all the outgoing light for this case we have the radiant exitance:

$$\begin{aligned}
L_r(\omega_r) &= f_r(\omega_r, \omega_i) L_i(\omega_i) \mu(\Omega_i) = f_r(\omega_r, \omega_i) E_i \\
M_r &= \int_{H^2} L_r(\omega_r)\, d\mu(\omega_r) \\
&= \int_{H^2} f_r(\omega_r, \omega_i)\, d\mu(\omega_r) \\
&= E_i \int_{H^2} f_r(\omega_r, \omega_i)\, d\mu(\omega_r).
\end{aligned}$$

So radiant exitance is less than incident irradiance exactly when

$$\int_{H^2} f_r(\omega_r, \omega_i)\, d\mu(\omega_r) < 1.$$

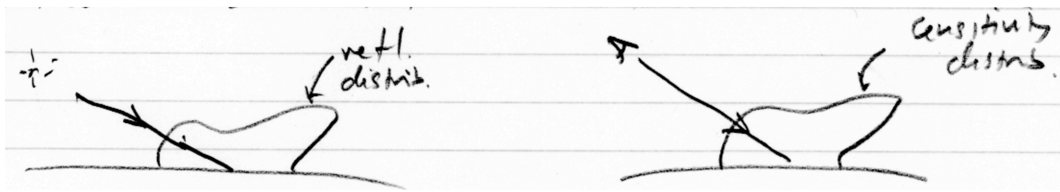
Here is a less hand-wavey proof that this guarantees energy conservation for arbitrary incident distributions:

$$\begin{aligned}
L_r(\omega_r) &= \int_{H^2} f_r(\omega_r, \omega_i)\, L_i(\omega_i)\, d\mu(\omega_i) \\
M_r &= \int_{H^2} L_r(\omega_r)\, d\mu(\omega_r) \\
&= \int_{H^2} \int_{H^2} f_r(\omega_r, \omega_i)\, L_i(\omega_i)\, d\mu(\omega_i)\, d\mu(\omega_r) \\
&= \int_{H^2} L_i(\omega_i) \left[\int_{H^2} f_r(\omega_r, \omega_i)\, d\mu(\omega_r) \right] d\mu(\omega_i) \\
&< \int_{H^2} L_i(\omega_i)\, d\mu(\omega_i) = E_i
\end{aligned}$$

Reciprocity This is part of a larger principle of reversibility of light transport paths, known as Helmholtz reciprocity or duality. In the context of the BRDF, the implication is that BRDFs are invariant with respect to swapping their arguments. That is:

$$f_r(\omega_1, \omega_2) = f_r(\omega_2, \omega_1).$$

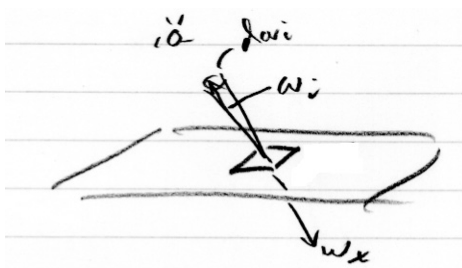
A physical interpretation is that the sensitivity distribution with the observer at a given position is the same as the reflected light distribution with the source at the same position.



This principle is true in reality, making it a great sanity-checking tool for measurements, and we try to ensure the BRDF models we invent are reciprocal, though most rendering systems will still work with non-reciprocal BRDFs.

2.4 BTDF and BSDF

I have so far only talked in detail about the BRDF, but this is only half of the function—the BSDF—that I promised to talk about. The other half is the BTDF, and there is really nothing new at all: its definition is identical to the BRDF, but without the constraint that the two vectors are on the same side of the surface. The BTDF is radiance over irradiance, just like the BRDF.



Nomenclature note: The R stands for "reflectance"; the T stands for "transmittance." The "-ance" means "per unit input." I'm not sure why we don't tend to use the word "scatterance" for the BSDF, which is used in some other fields; we tend to let the S stand for "scattering."