CS 5220: Parallel Graph Algorithms

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Graphs

Mathematically: $G = (V, E)$ where $E \subset V \times V$

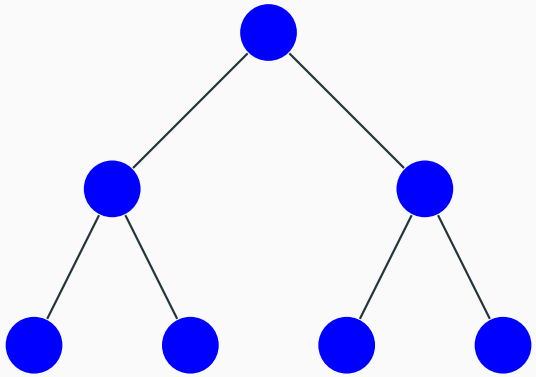
- Convention: $|V| = n$ and $|E| = m$
- May be directed or undirected
- May have weights $w_V : V \to \mathbb{R}$ or $w_E : E :\to \mathbb{R}$
- May have other node or edge attributes as well
- Path is $[\,(u_i, u_{i+1})\,]_{i=1}^{\ell} \in E^*$, sum of weights is length
- Diameter is max shortest path length between any $s, t \in V$

Generalizations:

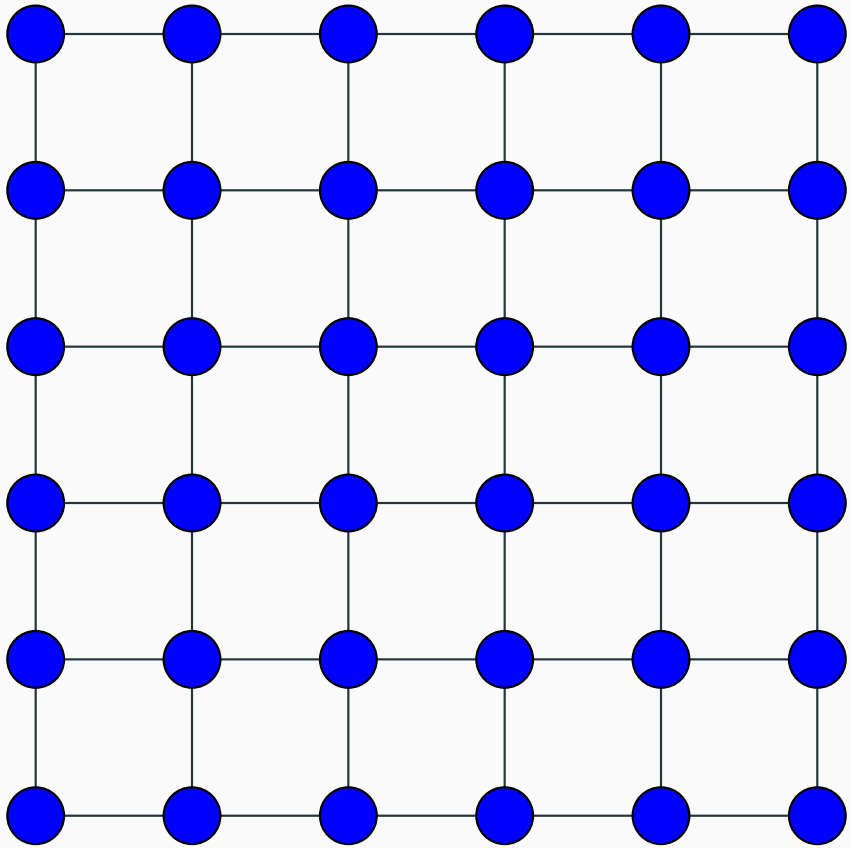
- Hypergraph (edges in V^d)
- Multigraph (multiple copies of edges)

Types of graphs

Types of graphs



Types of graphs



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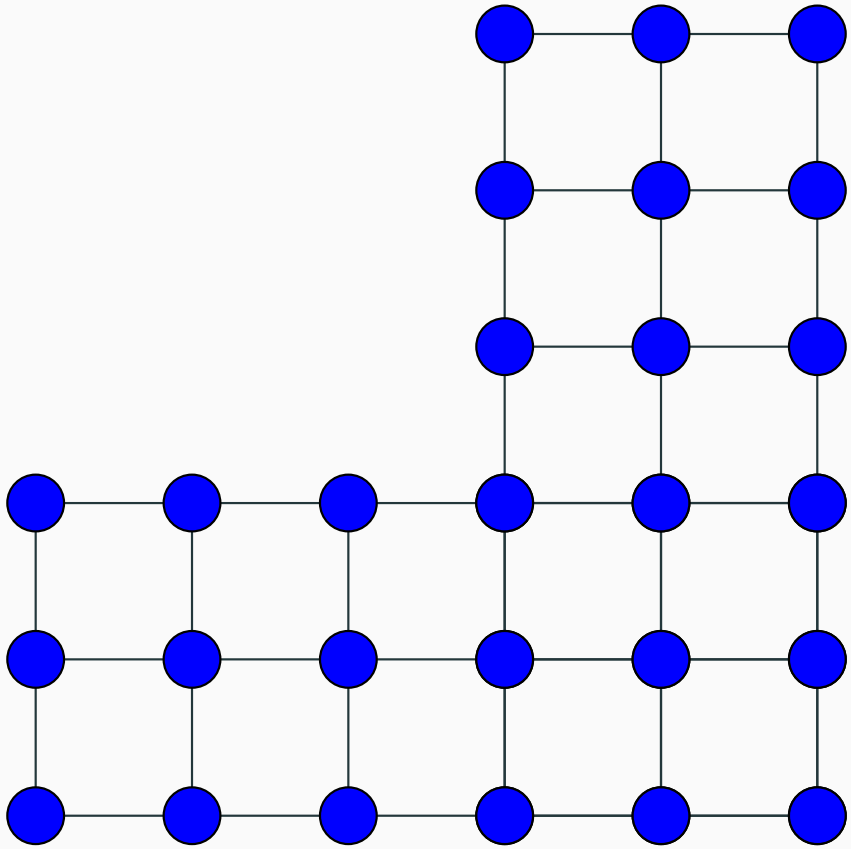
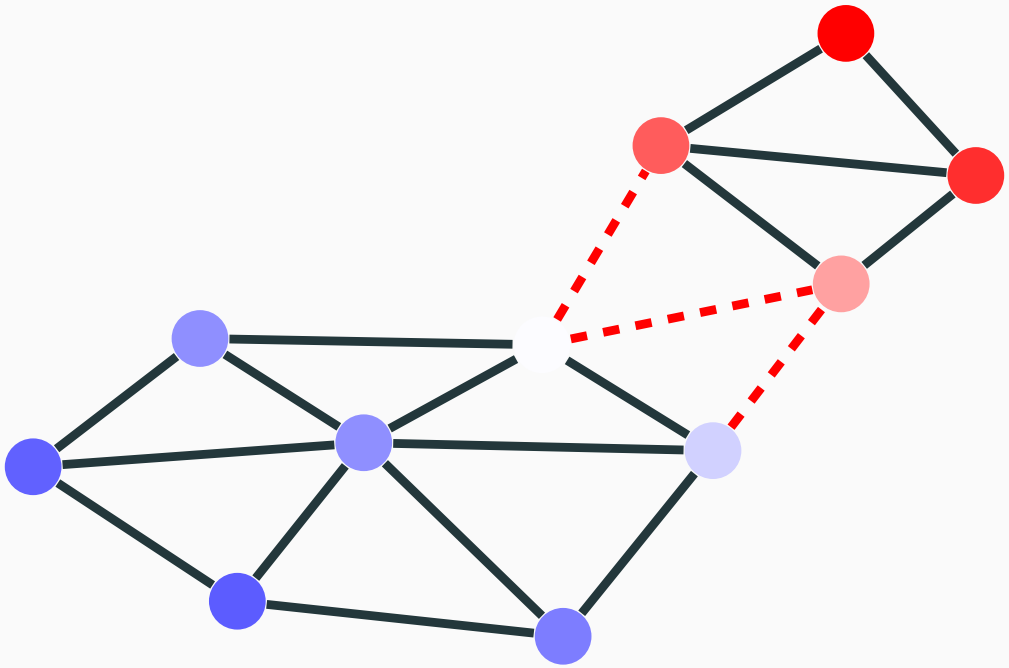
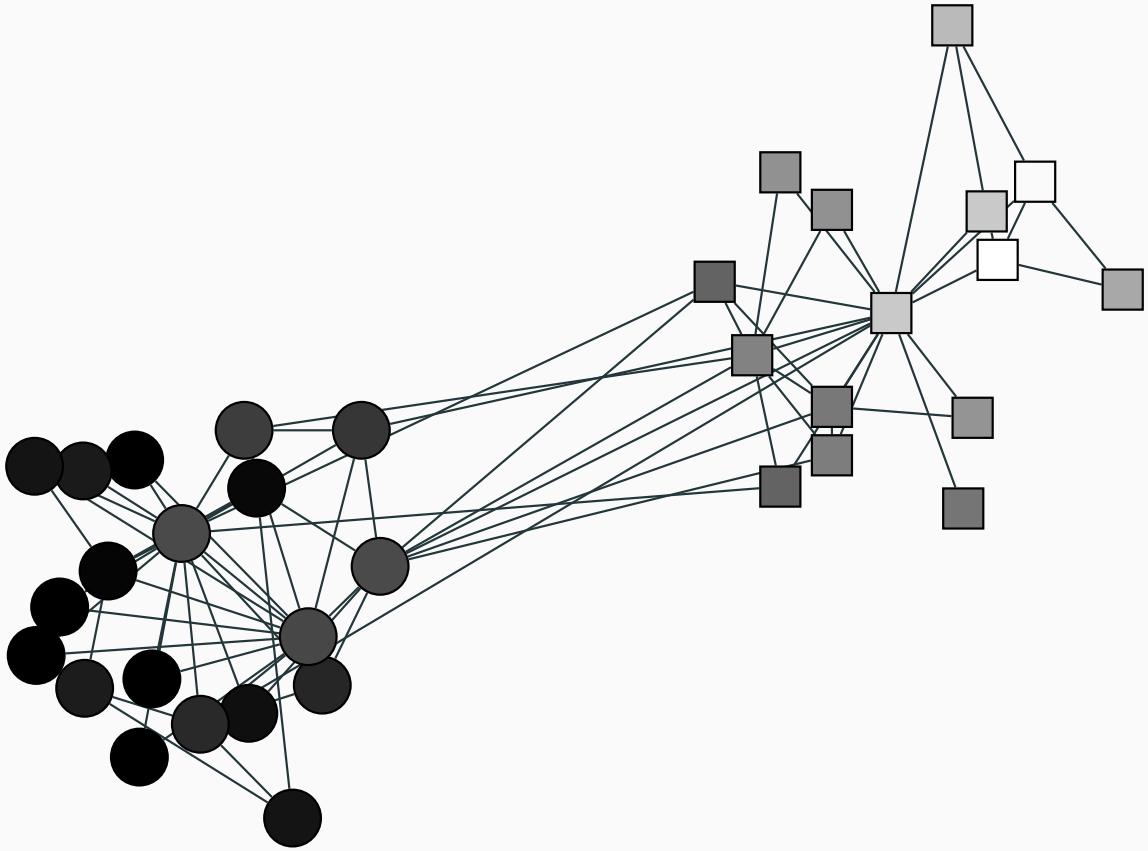


Types of graphs



Types of graphs



Types of graphs

Many possible structures:

- Lines and trees
- Completely regular grids
- Planar graphs (no edges need cross)
- Low-dimensional Euclidean
- Power law graphs
- ...

Algorithms are not one-size-fits-all!

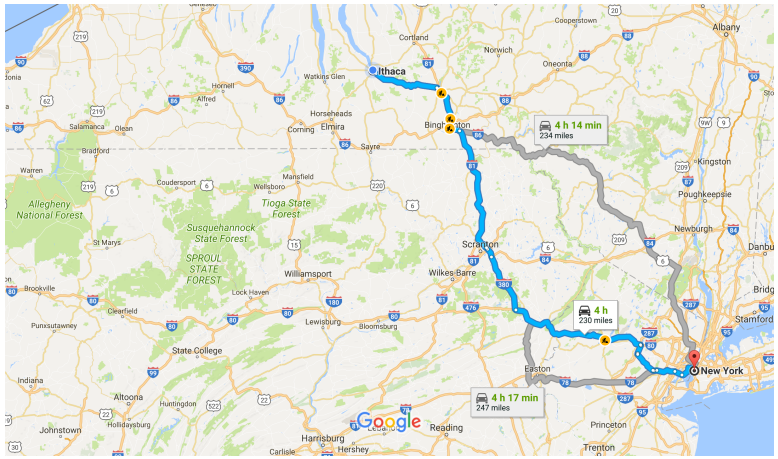
Ends of a spectrum

	Planar	Power law
Vertex degree	Uniformly small	$P(\deg = k) \sim k^{-\gamma}$
Radius	$\Omega(\sqrt{n})$	Small
Edge separators	$O(\sqrt{n})$	nothing small
Linear solve	Direct OK	Iterative
Prototypical apps	PDEs	Social networks

Calls for different methods!

Applications: Routing and shortest paths

Google Maps Ithaca, New York to New York Drive 230 miles, 4 h

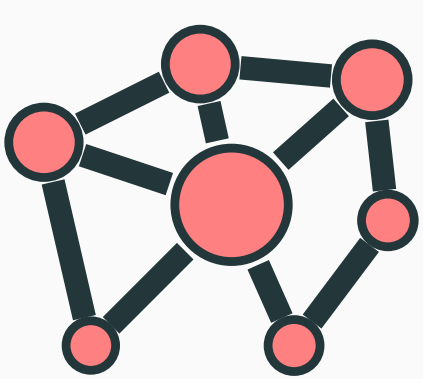


via I-81 S and I-80 E — 4 h
Fastest route now due to traffic conditions — 230 miles
⚠ This route has tolls.

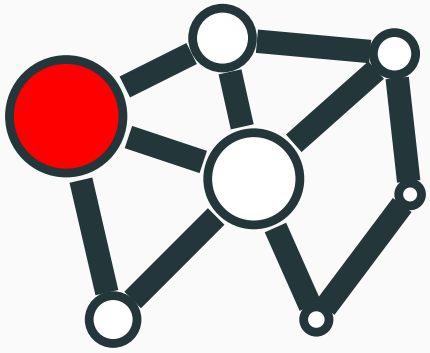
via NY-17 E — 4 h 14 min
234 miles

via I-81 S and I-78 E — 4 h 17 min
247 miles

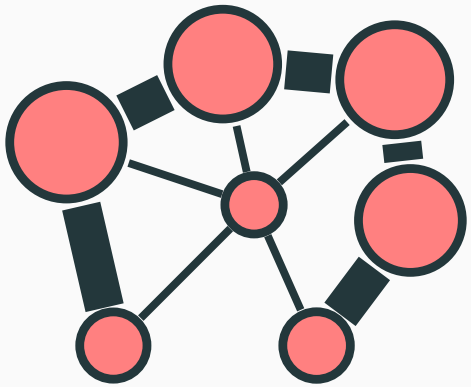
Applications: Traversal, ranking, clustering



Unweighted



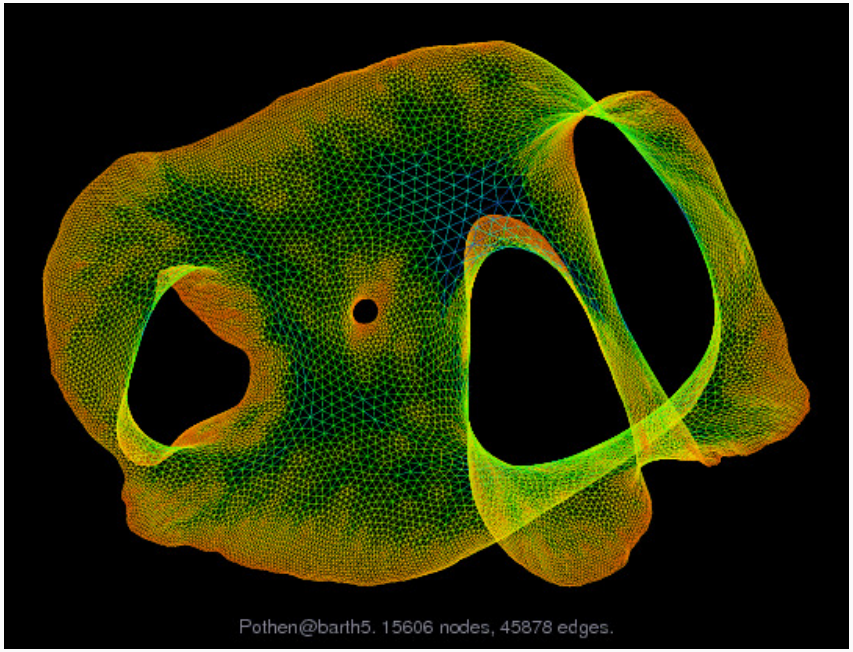
Node weighted



Edge weighted

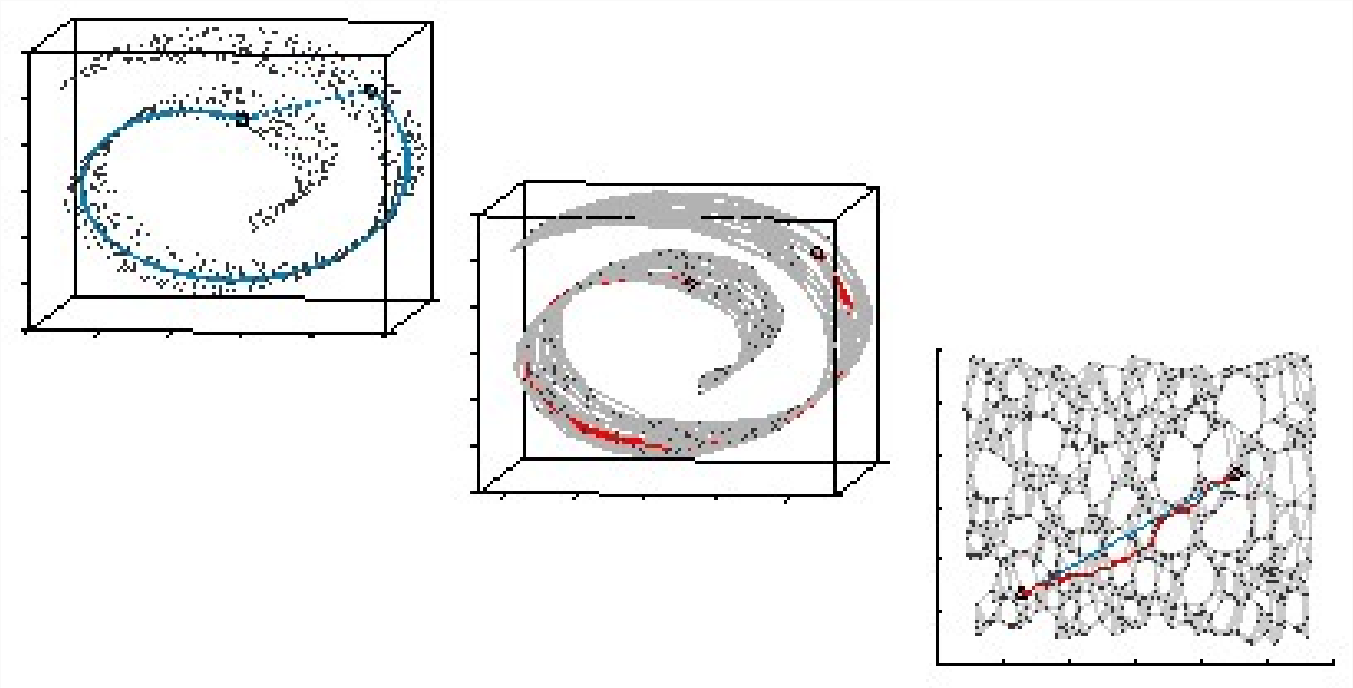
- Web crawl / traversal
- PageRank, HITS
- Clustering similar documents

Applications: Sparse solvers



- Ordering for sparse factorization
- Partitioning
- Graph coarsening for AMG
- Other preconditioning ops...

Applications: Dimensionality reduction

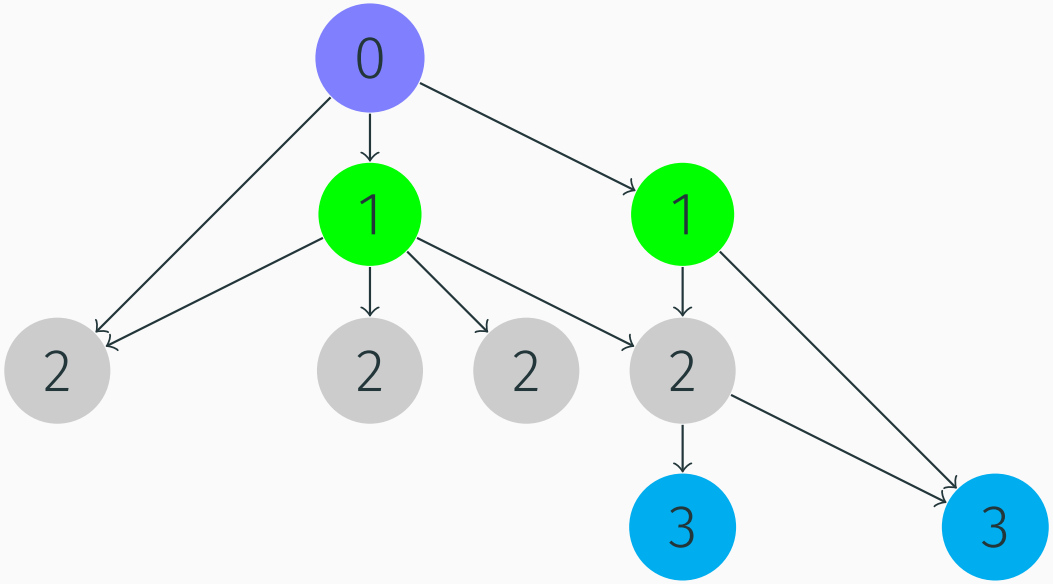


Common building blocks

- Traversals
- Shortest paths
- Spanning tree
- Flow computations
- Topological sort
- Coloring
- ...

... and most of sparse linear algebra.

Over-simple models



Let t_p = idealized time on p processors

- t_1 = work
- t_∞ = span (or depth, or critical path length)

One implication

Don't bother with parallel DFS! Span is $\Omega(n)$.

Let's spend a few minutes on more productive algorithms...

Parallel BFS

Simple idea: parallelize across frontiers

- Pro: Simple to think about
- Pro: Lots of parallelism with small radius?
- Con: What if frontiers are small?

Parallel BFS: Ullman-Yannakakis

Assuming a high-diameter graph:

- Form set S with start + random nodes, $|S| = \Theta(\sqrt{n} \log n)$ — long shortest paths must go through S with high prob
- Take $\sqrt{n}$ steps of BFS from each seed in S
- Form aux weighted graph for distances between seeds
- Run all-pairs shortest path on aux graph

OK, but what if diameter is not large?

LA take

- Indicate frontier at each stage by x
- $x' = A^T x$ (multiply=select, add=min)

Parallel BFS: LA perspective

Key ideas:

- At some point, switch from top-down expanding frontier ("are you my child?") to bottom-up checking for parents ("are you my parent?")
- Use 2D blocking of adjacency
- Temporally partition work: vertex processed by at most one processor at a time, cycle processors ("systolic rotation")

Together gives state-of-art performance. But...

Single-source shortest path

Classic algorithm: Dijkstra

- Dequeue closest point from frontier and expand frontier
- Update priority queue of distances (can be done in parallel)
- Repeat

Or run serial Dijkstra from different sources for APSP.

Alternate idea: label correcting

Initialize $d[u]$ with distance over-estimates to source

- $d[s] = 0$
- Repeatedly relax $d[u] := \min_{(v,u) \in E} d[v] + w(v, u)$

Converges (eventually) as long as all nodes visited repeatedly, updates are atomic. If serial sweep in a consistent order, call it Bellman-Ford.

Single-source shortest path: Δ-stepping

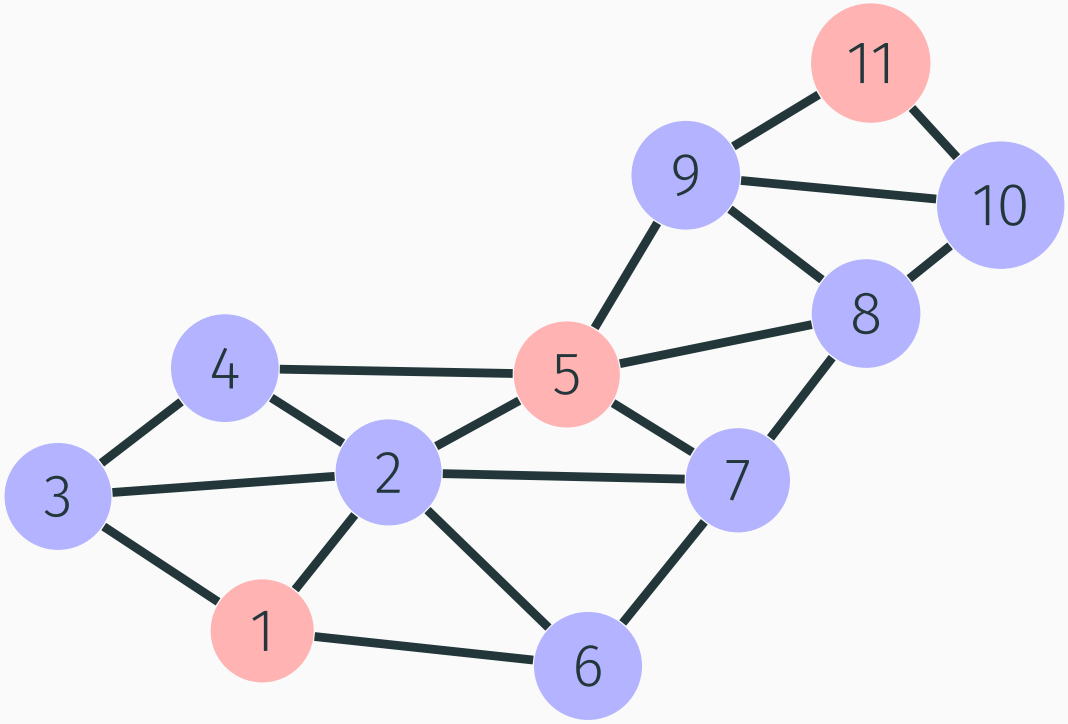
Alternate approach: *hybrid* algorithm

- Process a “bucket” at a time
- Relax “light” edges (weight < Δ) which might add to current bucket
- When bucket empties, relax “heavy” edges a la Dijkstra

Maximal independent sets

- $S \subset V$ *independent* if none are neighbors.
- *Maximal* if no others can be added and remain independent.
- *Maximum* if no other maximal independent set is bigger.
- Maximum is NP-hard; maximal is easy in one processor

Simple greedy MIS



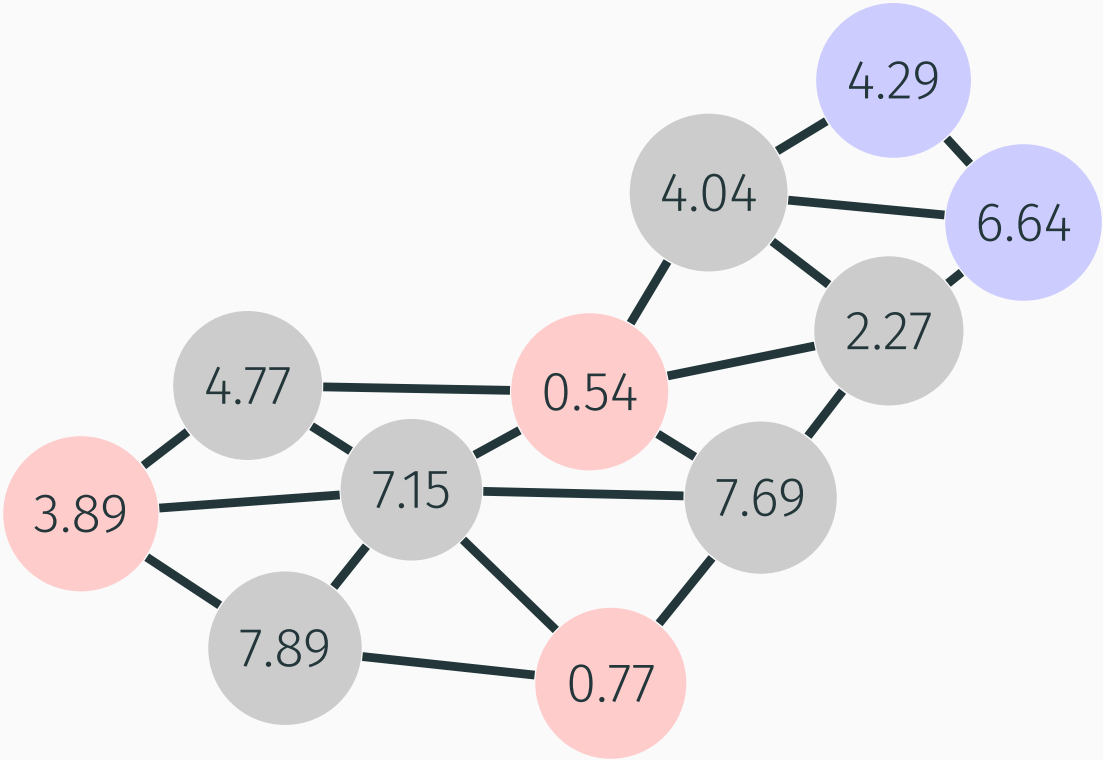
- Start with S empty
- For each $v \in V$ *sequentially*, add v to S if possible.

Luby's algorithm

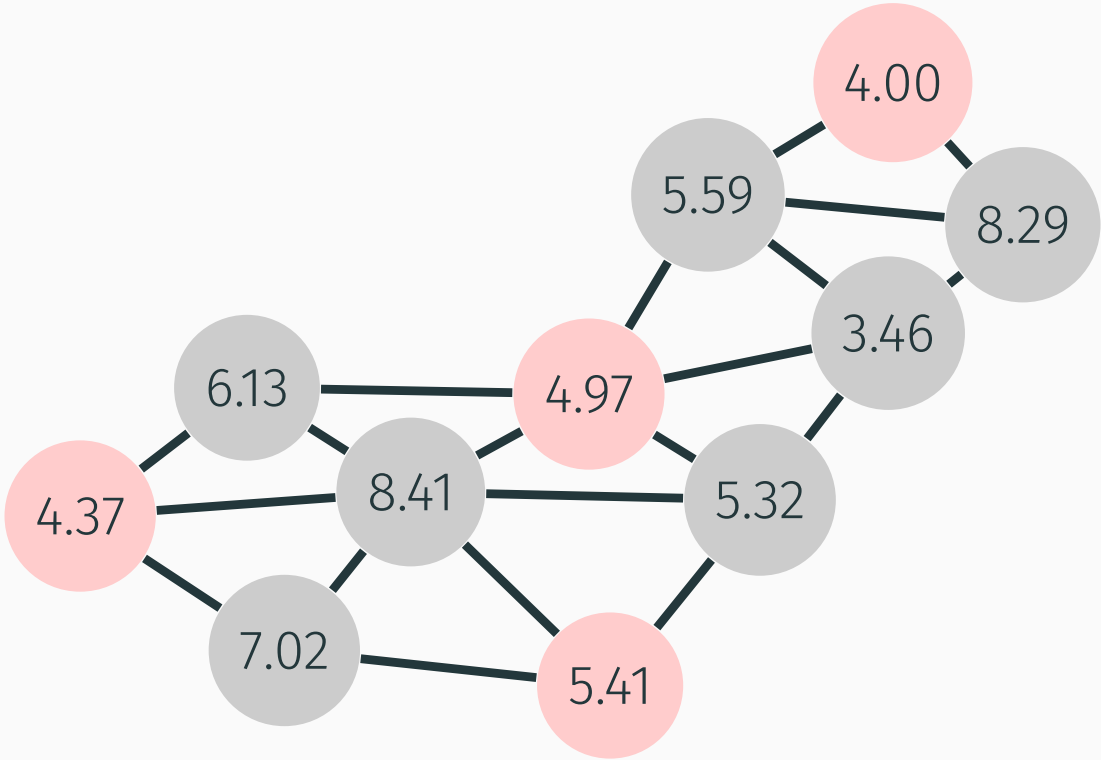
- Init $S := \emptyset$
- Init candidates $C := V$
- While $C \neq \emptyset$
  - Label each v with a random $r(v)$
  - For each $v \in C$ in parallel, if $r(v) < \min_{\mathcal{N}(v)} r(u)$
    - Move v from C to S
    - Remove neighbors from v to C

Very probably finishes in $O(\log n)$ rounds.

Luby’s algorithm (round 1)



Luby’s algorithm (round 1)



A fundamental problem

Many graph ops are

- Computationally cheap (per node or edge)
- Bad for locality

Memory bandwidth as a limiting factor.

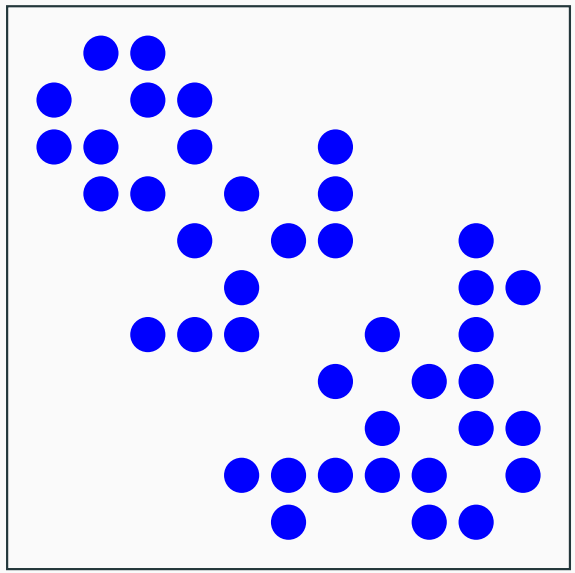
Big data?

Consider:

- 323 million in US (fits in 32-bit int)
- About 350 Facebook friends each
- Compressed sparse row: about 450 GB

Topology (no metadata) on one big cloud node...

Graph representation: Adjacency matrix



Pro: efficient for dense graphs

Con: wasteful for sparse case...

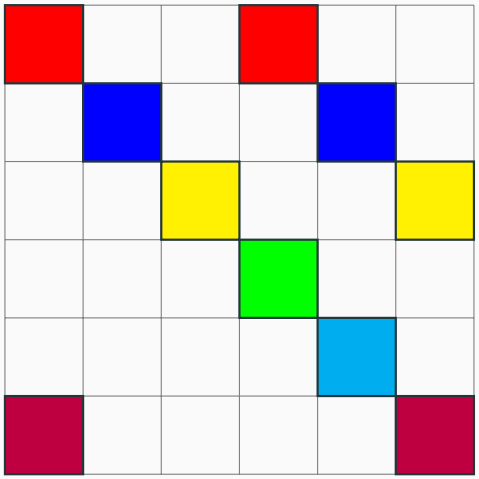
Graph representation: Coordinate

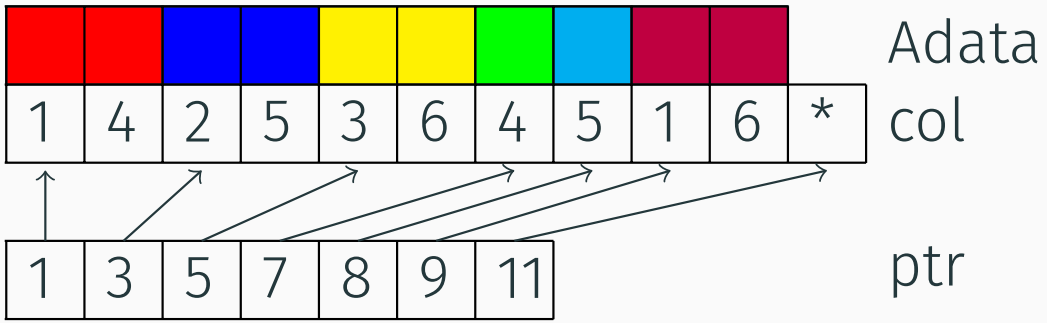
- Tuples: (i, j, w_{ij})
- Pro: Easy to update
- Con: Slow for multiply

Graph representation: Adj list

- Linked lists of adjacent nodes
- Pro: Still easy to update
- Con: May cost more to store than coord?

Graph representations: CSR





Pro: traversal? Con: updates

Graph representations: implicit

- Idea: Never materialize a graph data structure
- Key: Provide traversal primitives
- Pro: Explicit rep'n sometimes overkill for one-off graphs?
- Con: Hard to use canned software (except NLA?)

Graph algorithms and linear algebra

- Looks like LA
  - Floyd-Warshall
  - Breadth-first search?
- Really is standard LA
  - Spectral partitioning and clustering
  - PageRank and some other centralities
  - “Laplacian Paradigm” (Spielman, Teng, others…)

Graph algorithms and linear algebra

Semirings have $\oplus$ and $\otimes$ s.t.

- Addition is commutative+associative with an identity 0
- Multiplication is associative with identity 1
- Both are distributive
- $a \otimes 0 = 0 \otimes a = 0$
- But no subtraction or division

Technically have *modules* (vs vector spaces) over semirings

Graph algorithms and linear algebra

Example: min-plus

- $\oplus = \min$ and additive identity $0 \equiv \infty$
- $\otimes = +$ and multiplicative identity $1 \equiv 0$
- Useful for breadth-first search (on board)

Graph BLAS

`http://www.graphblas.org/`

- Provisional API as of late May 2017
- (Opaque) internal sparse matrix data structure
- Allows operations over misc semirings

Graph frameworks

Several to choose from!

- Pregel, Apache Giraph, Stanford GPS, ...
- GraphLab family
  - GraphLab: Original distributed memory
  - PowerGraph: Tuned toward “natural” (power law) networks
  - GraphChi: *Chi*huahua – shared memory vs distributed
- Outperformed by Galois, Ligra, BlockGRACE, others
- But... programming model was easy

Graph frameworks

- “Think as a vertex”
  - Each vertex updates locally
  - Exchanges messages with neighbors
  - Runtime actually schedules updates/messages
- Message sent at super-step S arrives at $S + 1$
- Looks like BSP

At what COST?

"Scalability! But at what COST?"
McSherry, Isard, Murray
HotOS 15

You can have a second computer once you've shown you know how to use the first one.
– Paul Barham (quoted in intro to HotOS15 paper)

- COST = Configuration that Outperforms a Single Thread
- Observation: many systems have unbounded COST!