Brouwer's Fixedpoint Theorem in Intuitionistic Mathematics

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Abstract

We present a proof of Brouwer's fixedpoint theorem in intuitionistic mathematics. Unlike most constructive proofs of the theorem it does not use Sperner's lemma. Instead, we first reduce the theorem to the no-retraction theorem and then use a constructive version of a proof by Karol Sieclucki that there is no retraction $|K| \to |\partial K|$ from a n-dimensional complex to its boundary. In our constructive version of this theorem we use *rational cubical complexes* rather than arbitrary simplicial complexes. We define the the polyhedron $|K|$ of a cubical complex K to be the *stable union* of the cubes in K. We prove that a stable union of finitely many compact spaces is compact and this allows us to conclude that $|K|$ is compact. In intuitionistic mathematics we may then derive a strong version of the fixedpoint theorem using the fact that all functions from a compact metric space to another metric space are uniformly continuous. Our proof has been completely checked using the proof assistant Nuprl. Nuprl's type theory is fully intuitionistic because it includes free choice sequences and has rules for bar induction and a continuity principle for numbers. Our argument could also be carried out in the system BISH of Bishop's school by proving the theorem only for uniformly continuous functions. Some additional work would be needed to prove that certain homeomorphisms (e.g. between a ball and a cube) are uniformly continuous, so we have not made a formal proof in BISH. Our proof is of interest because it is akin to Brouwer's original proof which used simplicial methods, and because it shows how a statement with constructive content (viz. the existence of an approximate fixedpoint) can be reduced to a statement with no constructive content (the non-existence of a retraction). Also, our use of the stable union, which is crucial to this proof, seems to be novel, and hence, a contribution to the tool kit of constructive mathematics (in both the BISH and INT systems).

1 Preliminaries

The real numbers. A sequence of integers $x = x_1, x_2, x_3, \ldots$ generates the sequence of rational numbers $\frac{x_1}{2}, \frac{x_2}{4}, \ldots \frac{x_n}{2n}, \ldots$. If we restrict to just those sequences for which the resulting rational sequence is Cauchy with the fixed modulus of Cauchyness $1/n$ we get the *regularity condition*

$$\forall n, m.\ |n * x_m - m * x_n| \leq 2(n + m)$$

We define $\mathbb{R}$ to be the type of regular sequences $x \in \mathbb{N}^+ \to \mathbb{Z}$. The basic arithmetic and ordering relations on $\mathbb{R}$ are defined as in the book by Bishop and Bridges. We remind the reader that the minimum m of two real numbers x and y can not be defined by (if $x < y$ then x else y), because $x < y$ is not decidable for real numbers, but, instead, m is simply the real number where $m_i = \min(x_i, y_i)$. The ordering on reals satisfies $\neg(x < y) \Rightarrow y \leq x$. Real numbers x and y are *separated* (written $x \# y$) if $(x < y) \vee (y < x)$. Another property we will use often is the fact that, even though $x < z$ is undecidable in general, if $x < y$ then for any z, either $x < z$ or $z < y$.

Sets, equality, and equivalence Bishop stated that to define a *set* one specifies a collection of things together with an equivalence relation on them– to be used for their equality relation. In classical mathematics based on ZFC, the collection of things X can be a set in ZFC and the equivalence relation $\equiv$ on X also a ZFC set. Then the "set" $\langle X, \equiv \rangle$ can be the quotient $X// \equiv$, the ZFC set of $\equiv$-equivalence classes of X. If $x, y \in X$ we then have two equality relations, $x = y$ and $[x] = [y]$ where $[x]$ is the $\equiv$-class of x, but both of these equality relations are the fundamental equality of ZFC. This is how the real numbers are defined in ZFC using either Dedekind cuts in the rationals or equivalence classes of Cauchy sequences of rationals.

In type theory, every type T has an equality relation (viz. an *equality type*) $x = y \in T$, but this may or may not correspond to the desired equivalence relation $\equiv$. For example, the equality relation $x = y \in \mathbb{R}$ is true just when, for all $i \in \mathbb{N}^+$, $x_i = y_i \in \mathbb{Z}$. This says that x and y are the same sequence of integers, but the equivalence relation that holds when x and y converge to the same real number turns out to be

$$x \equiv y \Leftrightarrow \forall i{:}\mathbb{N}^+.\ |x_i - y_i| \leq 4$$

In a type theory, like Nuprl's, where quotient types exist we could still form the quotient type $\mathbb{R}// \equiv$. But the problem with quotients (in ZFC or in type theory) is that in order to define functions on the quotient one must show that the function does not depend on the choice of representative of the equivalence class. This may not be possible unless one can use the axiom of choice to fix a "canonical" representative of each equivalence class. Such a canonical choice is not possible, constructively, for the equivalence classes of real numbers. Thus Bishop says that in constructive mathematics the use of equivalence classes is "either pointless or incorrect".

Therefore we will usually have two distinct relations on a type X, the equality $x = y \in X$ and an equivalence relation $\equiv$ on X. The pair $\langle X, \equiv \rangle$ is sometimes called a *setoid*. Following Bishop we call a member f of the type $X \to Y$ an *operation* on X (to type Y), even though the type $X \to Y$ is usually called a function type. For setoids $\langle X, \equiv_X \rangle$ and $\langle Y, \equiv_Y \rangle$ an operation $f \in X \to Y$ respects the equivalence relations if

$$\forall x_1, x_2 \colon X.\ x_1 \equiv_X x_2 \Rightarrow f(x_1) \equiv_Y f(x_2)$$

In this case we say that f is a *function* from $\langle X, \equiv_X \rangle$ to $\langle Y, \equiv_Y \rangle$ and write $f \in \mathrm{FUN}(X \to Y)$ (where the equivalence relations on X and Y are implicit, and usually clear from context).

In everyday practice of constructive mathematics we usually write $x = y$ when we really mean $x \equiv y$ and $f \in X \to Y$ when we really mean $f \in \mathrm{FUN}(X \to Y)$ but in this paper we will use $=$ only for equality in a type and $X \to Y$ only for operations. Thus, we will usually write $x \equiv y$ and $f \in \mathrm{FUN}(X \to Y)$.

Stable propositions. For any proposition P, $\neg\neg(P \vee \neg P)$ is provable. Also, if $P \Rightarrow Q$ then $(\neg\neg P) \Rightarrow (\neg\neg Q)$.

P is *decidable* if $P \vee \neg P$. P is *stable* if $(\neg\neg P) \Rightarrow P$. It is easy to see that if P is decidable then P is stable. Any negation $\neg Q$ is stable. If $\forall x \colon X.\ \mathrm{Stable}(P(x))$ then $\mathrm{Stable}(\forall x \colon X.\ P(x))$.

Lemma 1. *(Stable cases rule) If P is stable then for any proposition Q, P follows from $Q \Rightarrow P$ and $(\neg Q) \Rightarrow P$.*

Proof. Because P is stable, it is enough to prove $\neg\neg P$. Because $\neg\neg(Q \vee \neg Q)$ it is enough to prove $(Q \vee \neg Q) \Rightarrow P$. □

Because of this lemma, whenever we are proving a stable proposition we can use reasoning by cases as in classical logic. But we can only make finitely many such case splits. We can not simply assume that all propositions are decidable, nor can we assume strong forms of the axiom of choice. So many, but not all, classical proofs of stable propositions can be made constructive.

The ordering $x \leq y$ on real numbers is defined to be $\forall i\!:\!\mathbb{N}^+.\ x_i \leq y_i + 4$, so $x \leq y$ is stable. Similarly, $x \equiv y$, defined as $\forall i : \mathbb{N}^+.\ |x_i - y_i| \leq 4$, is stable, but $x < y$, defined as $\exists i : \mathbb{N}^+.\ x_i + 4 < y_i$, is not. In our proof of the no-retraction theorem we will often make use of the fact that the stable cases rule can use to prove a conclusion of the form $x \leq y$.

2 Compact metric spaces

An operation $d \in X \to X \to \mathbb{R}$ is a *metric* on X if

1. $d(x, x) \equiv 0$
2. $d(x, z) \leq d(x, y) + d(z, y)$

From these restrictions we can prove $d_X(x, y) \equiv d(y, x)$ and $d(x, y) \geq 0$. We also prove that $d(x, y) \equiv 0$ is an equivalence relation on X that we will write as $x \equiv_X y$ or just $x \equiv y$ when it is clear which metric space $\langle X, d\rangle$ is meant. Since the usual condition $d(x, y) = 0 \Leftrightarrow (x = y)$ is absent, our operation d is usually called a *pseudo-metric* (becoming a metric only on the $\equiv_X$-equivalence classes of X), but as discussed above, we do not want to use equivalence classes so we work with the setoid $\langle X, \equiv_X\rangle$.

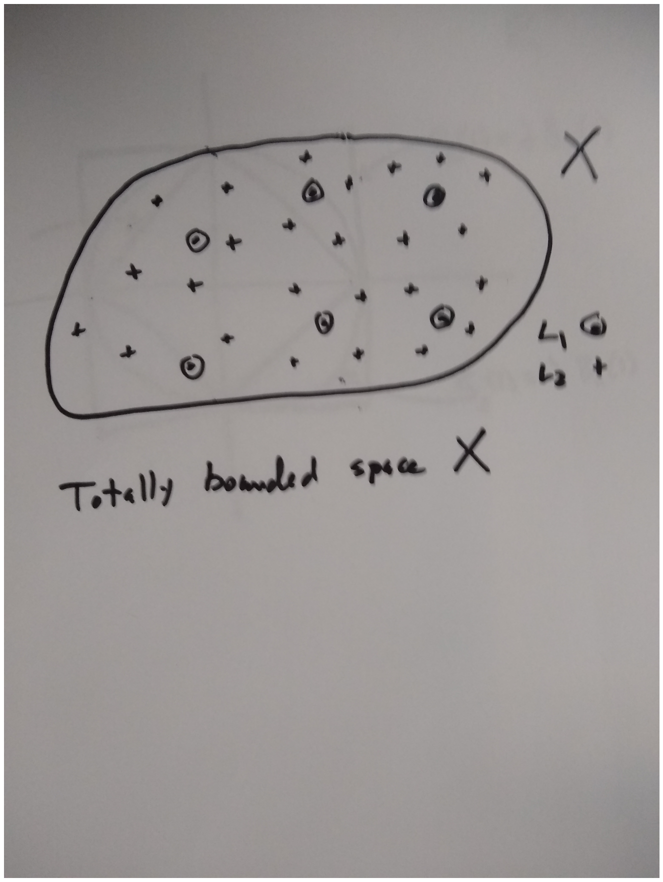
A sequence $s \in \mathbb{N}^+ \to X$ is *Cauchy* (in $\langle X, d\rangle$) if

$$\forall k\!:\!\mathbb{N}^+.\ \exists N\!:\!\mathbb{N}^+.\ \forall n, m \geq N.\ d(s_n, s_m) \leq 1/k$$

Sequence s is *converges to* x (written $\lim(s) = x$) if

$$\forall k\!:\!\mathbb{N}^+.\ \exists N\!:\!\mathbb{N}^+.\ \forall n \geq N.\ d(s_n, x) \leq 1/k$$

Metric space $\langle X, d\rangle$ is *complete* if every Cauchy sequence in $\langle X, d\rangle$ converges to some $x \in X$. The space is *totally bounded* if for every $k \in \mathbb{N}^+$ there is a finite list L_k of points in X such that for every $y \in X$ there is an $i < \|L_k\|$ with $d(y, L_k[i]) < 1/k$ (where $m = \|L_k\|$ is the length of the list $L_k = [L_k[0], \ldots L_k[m-1]]$). A metric space is *compact* if it is complete and totally bounded.



An operation $f \in X \to Y$ is *uniformly continuous* if

$$\forall k{:}\mathbb{N}^+.\ \exists m{:}\mathbb{N}^+.\ \forall x_1, x_2{:}X.\ d(x_1, x_2) \le 1/m \Rightarrow d'(f(x_1), f(x_2)) \le 1/k$$

If f is uniformly continuous then $f \in \mathrm{FUN}(X \to Y)$, so uniformly continuous operations can be called functions.

A basic construction involving a totally bounded metric space $\langle X, d\rangle$ is the construction, for any uniformly continuous function $f : X \to \mathbb{R}$, of the infimum $s = \inf(f(x) \mid x \in X)$. Given $n > 0$ we let $k > 0$ be such that $d(x_1, x_2) < 1/k \Rightarrow |f(x_1) - f(x_2)| < 1/2n$ and let $L = L_{2n}$ be the finite list given by the total boundedness property for X. Then the real number $s_n = \min(f(L[i]) \mid i < \|L\|)$ computes s within $1/n$ so we can take $s = \lim(s_n)$, since $\mathbb{R}$ is complete.

We next state, without proof, a fundamental theorem of intuitionistic mathematics that follows from the FAN theorem (which is proved using bar induction) and the continuity principle for numbers.

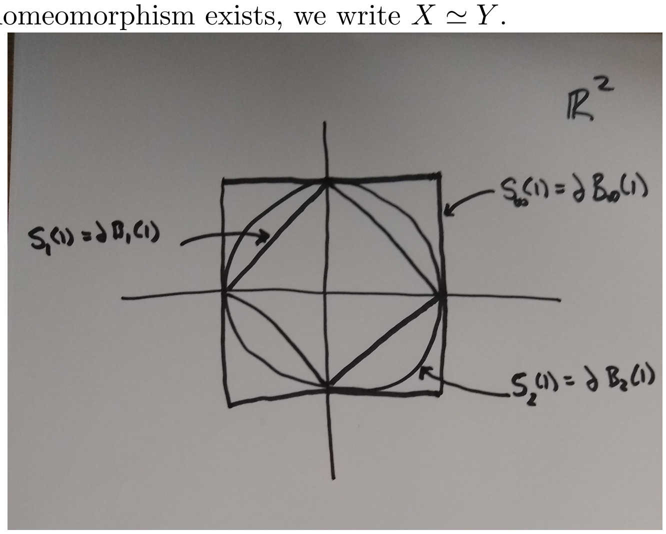
Theorem 1. *If $\langle X, d\rangle$ is a compact metric space, $\langle Y, d'\rangle$ is a metric space, and $f \in FUN(X \to Y)$ then f is* uniformly continuous*.*

We have proved this theorem in Nuprl. It is the only result of intuitionistic mathematics not provable in BISH that we will use in our proof of Brouwer's fixedpoint theorem. Thus, to carry out the rest of this paper in BISH, one must assume that given functions are uniformly continuous (rather than just FUN's) and prove that constructed functions are uniformly continuous. That extra work could be done and a weaker version of the fixedpoint theorem could be proved in BISH using our arguments, but we have not made such a formal proof.

Note that by Theorem 1 we can compute $\inf(f(x) \mid x \in X)$ for any $f \in \mathrm{FUN}(X \to \mathbb{R})$ when X is compact. This also means that when $A \subseteq X$ is compact and $x \in X$ then the distance $d(x, A) = \inf(d(x, a) \mid a \in A)$ exists and has the expected properties, in particular, $d(x, A) \equiv 0 \Leftrightarrow x \in A$.

Homeomorphic metric spaces. For any metric spaces X and Y, and function $f \in \mathrm{FUN}(X \to Y)$, Theorem 1 implies that f is uniformly continuous on compact subsets $A \subseteq X$. Thus, f satisfies Bishop's definition of a continuous function from X to Y. So we see that in intuitionistic mathematics we can take $f \in \mathrm{FUN}(X \to Y)$ to be our definition of a continuous function.

This simplifies the definition of homeomorphism. A pair $\langle f, g\rangle$ is a *homeomorphism* between metric spaces X and Y if $f \in \mathrm{FUN}(X \to Y)$, $g \in \mathrm{FUN}(Y \to X)$, and $\forall x\!:\!X.\ g(f(x)) \equiv x$ and $\forall y\!:\!Y.\ f(g(y)) \equiv y$. When a homeomorphism exists, we write $X \simeq Y$.



Metrics on $\mathbb{R}^k$. The *product metric* on $\mathbb{R}^k$ is $d_1(x,y) = \Sigma\{|x_i - y_i| \mid i < k\}$. The *Euclidean metric* is $d_2(x,y) = \sqrt{\Sigma\{(x_i - y_i)^2 \mid i < k\}}$, and the *max metric* is $d_\infty(x,y) = \max\{|x_i - y_i| \mid i < k\}$. All three of these metrics generate the same equivalence relation on $\mathbb{R}^k$ because all three satisfy $d(x,y) \equiv 0 \Leftrightarrow (\forall i < k.\ x_i \equiv y_i)$. The three metrics are related by

$$d_\infty(x,y) \leq d_2(x,y) \leq d_1(x,y) \leq k * d_\infty(x,y)$$

$\mathbb{R}^k$ is complete in the product metric, and hence, in all three metrics. For each of the metrics d_i, $i \in \{1, 2, \infty\}$ the corresponding *norm* is $\|x\|_i = d_i(x, 0)$, and the *ball* $B_i(r) = \{x : \mathbb{R}^k \mid \|x\|_i \leq r\}$.

Vectors $x, y \in \mathbb{R}^k$ are *separated* ($x \# y$) if $\exists i < k.\ x_i \# y_i$. For $i, j \in \{1, 2, \infty\}$ the operation $r^i_j(x) = \frac{\|x\|_i}{\|x\|_j}$ is defined when $x \# 0$ but can be extended to all of $\mathbb{R}^k$, by using limits in $\mathbb{R}^k$, so that $r^i_j(x) \equiv 1$ when $x \equiv 0$. Then the operations $x \mapsto r^i_j(x) * x$ and $x \mapsto r^j_i(x) * x$ are inverses and respect the equivalence relations so they are functions giving a homeomorphism from $\mathbb{R}^k$ to $\mathbb{R}^k$ that carries the ball $B_i(r)$ to the ball $B_j(r)$.

For real numbers $a \leq b$, the closed interval $[a, b]$ is compact. For $a, b \in \mathbb{R}^k$ the *real cube*[1] $\mathrm{Cube}(a, b)$ is $\{x : \mathbb{R}^k \mid \forall i < k.\ a_i \leq x_i \leq b_i\}$. The cube is *inhabited* if $\forall i < k.\ a_i \leq b_i$. It is straightforward to prove that an inhabited cube in $\mathbb{R}^k$ is compact in the product metric, and hence, in all three metrics. Since the ball $B_\infty(r)$ is the cube $\mathrm{Cube}(-r, r)$, it is compact when $r \geq 0$. Using the homeomorphisms discussed above, we prove that all the balls $B_i(r)$ are compact (for $r \geq 0$).

Stable union of metric spaces. If X is a metric space, $A \subseteq X$, and $B \subseteq X$ then the union of A and B is $A \cup B = \{x : X \mid (x \in A) \vee (x \in B)\}$. If both A and B are complete, the union $A \cup B$ may not be complete because to show that a given Cauchy sequence $s \in \mathbb{N}^+ \to (A \cup B)$ converges to a point in $A \cup B$ one would have to find either a sub-sequence s_A of s with all points in A or a sub-sequence s_B of s with points in B. This would require the ability to decide for an arbitrary function $p \in \mathbb{N}^+ \to 2$ whether for infinitely many n, $p(n) = 0$ or for infinitely many n, $p(n) = 1$. The continuity principle implies that such a decision depends on only finitely many values of $p(n)$, so it is impossible.

Another way to understand the problem is to consider the union of the two intervals $[0, 1]$ and $[1, 2]$. To prove that this union contains the interval $[0, 2]$ we would have to decide, for an arbitrary $z \in [0, 2]$ whether $z \leq 1$ or $1 \leq z$, and (by the continuity principle again) this can not be done.

[1] More properly k-orthotope, hyperrectangle, or box.

Definition 1. *The* stable union *of $A \subseteq X$ and $B \subseteq X$ is*

$$A \,\dot{\cup}\, B = \{x\!:\!X \mid \neg\neg((x \in A) \vee (x \in B))\}$$

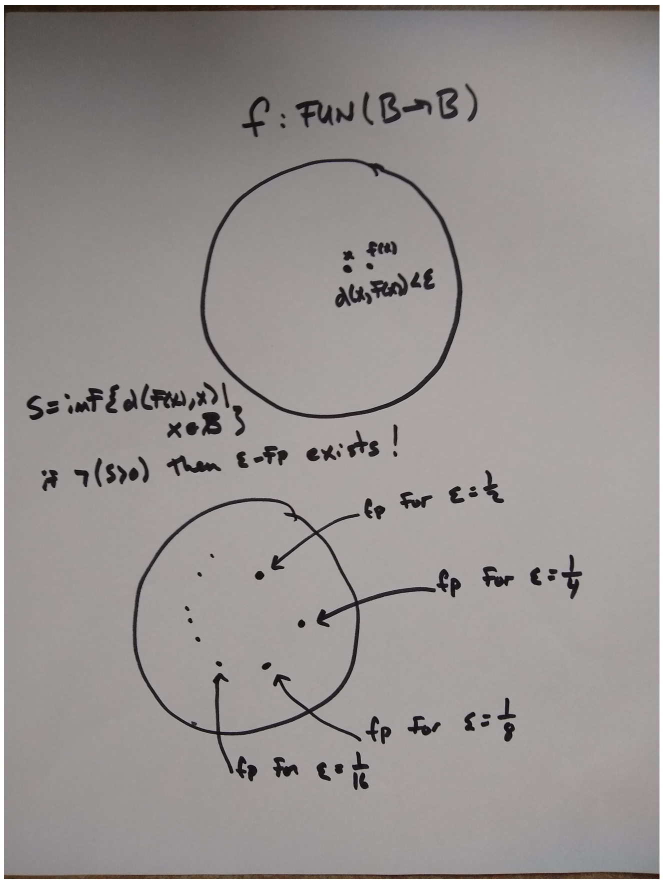
Theorem 2. *If $\langle X, d\rangle$ is a complete metric space, and $A \subseteq X$ and $B \subseteq X$ are compact metric spaces, then $A \,\dot{\cup}\, B$ is compact.*

Proof. To prove that $A \,\dot{\cup}\, B$ is complete we note that a Cauchy sequence $s \in \mathbb{N}^+ \to A \,\dot{\cup}\, B$ converges in $\langle X, d\rangle$ to some $x \in X$. We prove

$$\neg\neg(x \in A \vee x \in B)$$

Using the stable cases rule, we have either $0 < d(x, A)$ or $d(x, A) \equiv 0$. If $d(x, A) \equiv 0$ then $x \in A$ and we are done, so we may assume $d(x, A) > 1/k$ for some k. Similarly, we may assume $d(x, B) > 1/k'$ for some k'. But then we can prove False because $\exists N \in \mathbb{N}^+$ such that $\forall i > N.\ d(s_i, x) < 1/\max(k, k')$.

To prove that $A \,\dot{\cup}\, B$ is totally bounded, let $k > 0$ and let lists A_{2k} and B_{2k} witness the total boundedness of A and B for $2k$ and let L_k be the list A_{2k} appended to B_{2k}. Let x be any point $A \,\dot{\cup}\, B$. Then $x \in X$ and because $1/2k < 1/k$ we can decide, for any y in the list L_k whether $1/2k < d(x, y)$ or $d(x, y) < 1/k$. If all of these decisions result in $1/2k < d(x, y)$ then we prove False by cases on $x \in A$ or $x \in B$, since L_k includes both A_{2k} and B_{2k}. Hence, for some $y \in L$, $d(x, y) < 1/k$ which shows that L_k witnesses the total boundedness of $A \,\dot{\cup}\, B$ for k. □



3 Brouwer's fixedpoint theorem, Step One

Metric space $\langle X, d \rangle$ satisfies the *fixedpoint property* $\mathrm{FP}(X)$ if every *function* from X to itself has approximate fixpoints. Formally:

$$\forall f\!:\!\mathrm{FUN}(X \to X).\ \forall k\!:\!\mathbb{N}^+.\ \exists x\!:\!X.\ d(f(x), x) \leq 1/k$$

For a compact metric space X, if $\mathrm{FP}(X)$ and $X \simeq Y$, then $\mathrm{FP}(Y)$.

Theorem 3. *(Brouwer's fixedpoint theorem) For any $n \geq 0$ and $r \geq 0$, the ball $B(n,r) = \{x\!:\!\mathbb{R}^n \mid \|x\| \leq r\}$ has the fixedpoint property: $FP(B(n,r))$.*

To begin the proof, we first note that it does not matter which of the norms $\|x\|_1$, $\|x\|_2$, or $\|x\|_\infty$ is used for the ball because for a given $r \geq 0$ all three balls are compact and homeomorphic. Second, we may as well assume

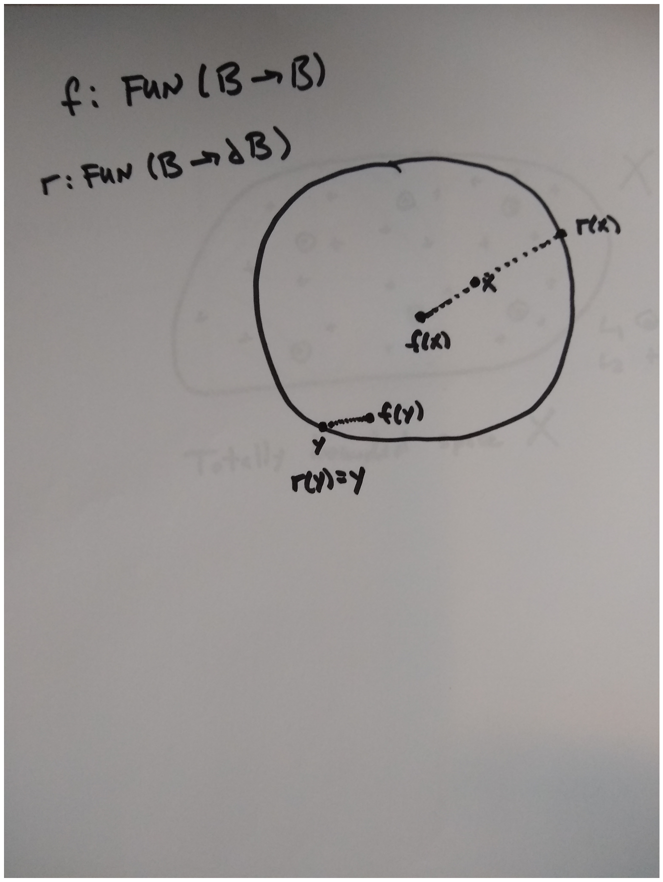
$r = 1$ because for a given $k \in \mathbb{N}^+$, we have either $0 < r$ or $r < 1/k$. In the latter case the origin 0 is an approximate fixedpoint. If $0 < r$ then the balls $B(n, r)$ and $B(n, 1)$ are homeomorphic. So it is enough to prove the fixedpoint property for the Euclidean unit ball $B(n)$.

Now, if $f \in \mathrm{FUN}(B(n) \to B(n))$ then by Theorem 1, f is uniformly continuous and hence, the infimum $s = \inf\{d(f(x), x) \mid x \in B(n)\}$ exists since $B(n)$ is compact. By the definition of the infimum, for any $\epsilon > 0$ there is an $x \in B(n)$ such that $d(f(x), x) < s + \epsilon$. We have $s \geq 0$ because $d(f(x), x) \geq 0$. If we can prove that $\neg(s > 0)$ then we will have $s \equiv 0$ and hence, for any $\epsilon > 0$ there exists $x \in B(n)$ such that $d(f(x), x) < \epsilon$; which is the fixedpoint property.

So we have already reduced the proof of Brouwer's fixedpoint theorem to proving a negation, viz. $\neg(s > 0)$. This seems somewhat paradoxical because the proof of a negation will have no constructive content, but the fixedpoint property asserts that something exists, so it does have constructive content. The solution to this paradox is that the constructive content of the fixedpoint theorem comes from the construction of the infimum $s = \inf\{d(f(x), x) \mid x \in B(n)\}$.

By examining the construction of the infimum, we see that to find an ϵ-fixedpoint for f we proceed as follows. For any finite list L of points in $B(n)$ we can test for each point $x \in L$ whether $\epsilon/2 < d(f(x), x)$ or $d(f(x), x) < \epsilon$ for each $x \in L$. This will either find an ϵ-fixedpoint of f or will show that $\epsilon/2 < d(f(x), x)$ for all $x \in L$. But if $\delta > 0$ is such that for points in $B(n)$, $d(x, y) \leq \delta \Rightarrow d(f(x), f(y)) \leq \epsilon/2$ and L is the list witnessing the total boundedness of $B(n)$ for δ, then the second possibility would show that $s > 0$. Thus, assuming that $\neg(s > 0)$ we will find the ϵ-fixedpoint in that list L.

This algorithm for finding the ϵ-fixedpoint can be simplifed because we don't really need to know what the modulus of uniform continutity δ is. Knowing only that δ exists, we can simply test the lists L_1, $L_2 \ldots L_k \ldots$ for the ϵ-fixedpoint. Since for some k we will have $1/k < \delta$, our search will eventually terminate. Because we can use rational points to witness the total boundedness of $B(n)$ we can simplify even further and simply test the *rational* points in $B(n)$ beginning with points with denominator 1 then denominator 2, denominator 3, etc.. Assuming $\neg(s > 0)$ this search will eventually terminate.



Reduction to the no-retraction theorem. The *boundary* $\partial B(n)$ of the ball $B(n)$ is $\{x : \mathbb{R}^n \mid \|x\| \equiv 1\}$. If $s > 0$ then for every $x \in B(n)$ we have $d(f(x), x) \geq s > 0$ so the vector $v = (x - f(x))$ from point $p = f(x)$ to the point x has positive norm $\|v\| > 0$.

The ray $p + t * v$ for $t \geq 0$ intersects $\partial B(n)$ when $\|p + t * v\| \equiv 1$, or, equivalently, when $(p + t * v) \cdot (p + t * v) = (p \cdot p) + 2(p \cdot v)t + (v \cdot v)t^2 = 1$. This is a quadratic equation $at^2 + bt + c = 0$ with $a = \|v\|^2 > 0$ and $c = (\|p\|^2 - 1) \leq 0$, so it can be solved with $t = (-b + \sqrt{b^2 - 4ac})/2a$. Then $(t \geq 0) \Leftrightarrow (\sqrt{b^2 - 4ac} \geq b) \Leftrightarrow (ac \leq 0)$ so t is non-negative.

We let $r(x) = f(x) + t * (x - f(x))$ where t is computed as above. Then $r \in \text{FUN}(B(n) \to \partial B(n))$. When $\|x\| = 1$ we have $t = 1$ and $r(x) \equiv x$, so r is a *retraction* from $B(n)$ to $\partial B(n)$.

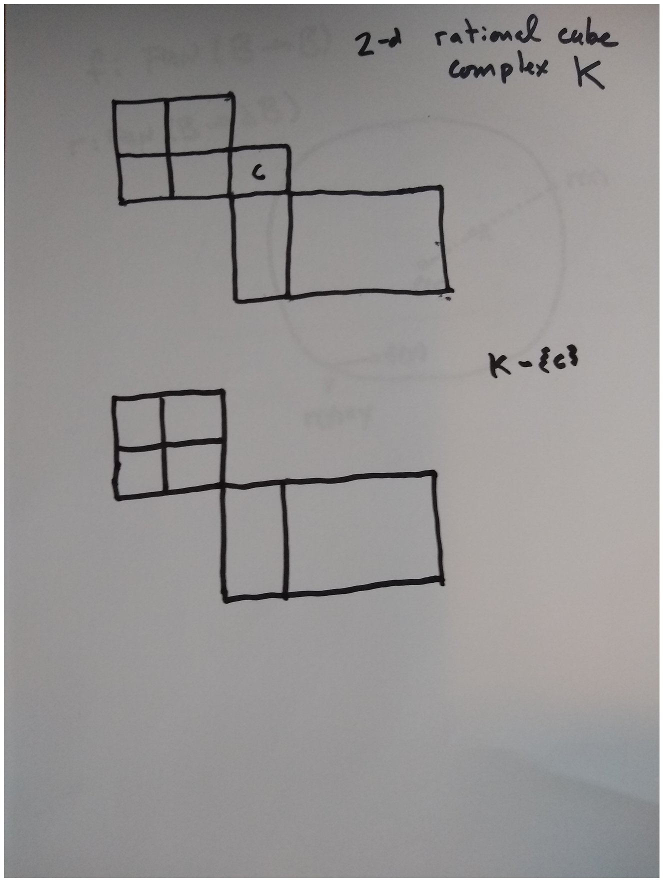
Once we prove that there is no retraction from $B(n)$ to $\partial B(n)$ then we will have proved $\neg(s > 0)$ and Theorem 3.

To prove the no-retraction theorem for $B(n)$ we will generalize the theorem to the non-existence of retractions $|K| \to |\partial K|$ for n-dimensional *rational cubical complexes* K. This is proved by induction on n using an argument adapted from one given by Karol Sieclucki.

4 Rational Cube Complexes

All vectors v, p, q, etc. will be in $\mathbb{R}^k$ for some fixed k. All the coordinates of a *rational vector* in $\mathbb{Q}^k$ are rational numbers. For rational numbers r and s we can decide $r = s$, $r < s$, and $r \leq s$. A *rational cube* $c = \mathrm{Cube}(a, b)$ is given by two rational vectors $a, b \in \mathbb{Q}^k$. Vector $p \in \mathbb{R}^k$ is in the cube c ($p \in c$) if $a_i \leq p_i \leq b_i$ for all $i < k$. So, cube c is *inhabited* if and only if $a_i \leq b_i$ for all $i < k$.

The i^{th} interval $[a_i, b_i]$ has dimension 1 if $a_i < b_i$ and has dimension 0 if $b_i \leq a_i$. The dimension of cube c is the sum of the dimensions of each of its intervals, so $0 \leq \dim(c) \leq k$. The *faces* of interval $[a, b]$ are the two 0-dimensional intervals $[a, a]$ and $[b, b]$ and interval $[a, b]$ itself. A cube f is a *face* of cube c if, for all $i < k$, the i^{th} interval of f is a face of the i^{th} interval of c. We write $f \leq c$ when f is a face of c. An n-dimensional cube c has exactly $2n$ faces with dimension $n-1$, obtained by replacing just one of the 1-dimensional intervals of c by one of its two 0-dimensional faces.

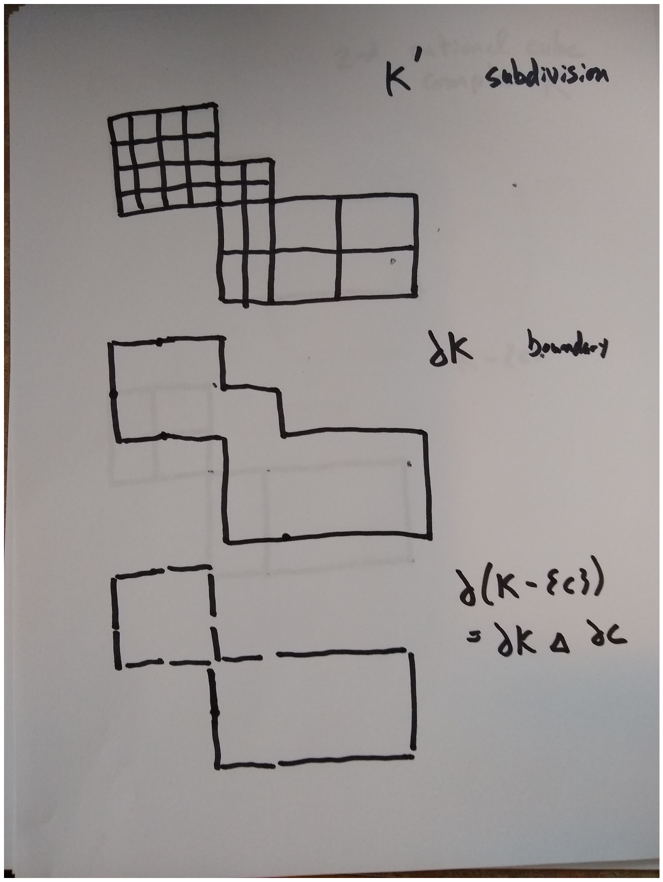


An n-dimensional *rational cube complex* (n-dim complex, for short) is a finite list K of rational cubes, with no repeats, such that each cube $c \in K$ is inhabited and has dimension n, and for any two cubes c and d in K, if they have a point in common (i.e. they overlap) then there is a common face $f \leq c$, $f \leq d$ such that the intersection of c and d is exactly f. So, we say that the cubes in K overlap only along faces.

The *boundary*, ∂K, of the n-dim complex K is the list of $n-1$ dimensional cubes f that are faces of an *odd number* of cubes in K. Then ∂K is an $n-1$-dimensional complex.

A cube d is a *half-cube* of cube c if for each interval $[a_i, b_i]$ of c, the corresponding interval of d is either $[a_i, \frac{a_i+b_i}{2}]$ or $[\frac{a_i+b_i}{2}, b_i]$. Each n-dimensional cube c has 2^n half cubes. The *sub-division* K' of complex K is the list of all the half cubes of K.

In the following lemmas, K is an n-dimensional rational cube complex.



Lemma 2. $\partial(\partial K) = \emptyset$.

Proof. Suppose, towards a contradiction, that $x \in \partial(\partial K)$. Then $\dim(x) = n-2$. Let $R(f,c) \Leftrightarrow (x \leq f \leq c \in K \wedge \dim(f) = n-1)$. We count the number of pairs $\langle f, c\rangle$ such that $R(f,c)$ in two ways. For a given n-dimensional c, the number of f satisfying $R(f,c)$ is even, because it is either two or zero. So the total number of pairs is even. On the other hand, for a given $n-1$-dimensional f, the number of c satisfying $R(f,c)$ can be either even or odd, but if it is odd, then $f \in \partial K$ and because $x \in \partial(\partial K)$ the number of such f must also be odd. Thus the total number of pairs is the sum of some even numbers plus and odd number of odd numbers, so it is odd. This gives a contradiction and proves the lemma. □

Lemma 3. $\partial(K') = (\partial K)'$.

Proof. This is geometrically obvious but because the definition of the boundary is combinatorial in nature it requires proof. The key observation is that if h is a half cube of c and f is an $n-1$-dimensional face of h then either f is half of an $n-1$-dimensional face of c or else f contains the center point of c and there is exactly one other half cube h' of c with $f \leq h'$. Since cubes in K can not overlap at their center points, this proves that faces of an odd number of half cubes must be the halves of faces of an odd number of cubes. This is a sketch of the (somewhat tedious) proof that we carried out formally in Nuprl. □

Lemma 4. $c \in K \Rightarrow \partial(K \setminus \{c\}) = (\partial K \bigtriangleup \partial\{c\})$.

Proof. We are using $L \bigtriangleup L'$ for the *symmetric difference* of the lists L and L'. Let f be an $n-1$-dimensional cube, then $f \in \partial\{c\} \Leftrightarrow (f \leq c)$. If $\neg(f \leq c)$ then, clearly, $(f \in \partial(K \setminus \{c\})) \Leftrightarrow (f \in \partial K)$. If $f \leq c$ then $(f \in \partial(K \setminus \{c\})) \Leftrightarrow \neg(f \in \partial K)$ because the parity of the number of cubes of which f is a face changes.

□

Lemma 5. *length*$(\partial(K))$ *is even.*

Proof. By induction on length(K). length$(K) = 0 \Rightarrow$ length$(\partial(K)) = 0$. If $c \in K$ then length$(\partial\{c\}) = 2n$ is even and, by induction, length$(\partial(K \setminus \{c\}))$ is even. By Lemma 4, $\partial(K \setminus \{c\}) = (\partial K \bigtriangleup \partial\{c\})$, so length$(\partial(K))$ is even. □

Definition 2. *The* polyhedron*, $|K|$, of the complex K is the stable union*

$$\{p : \mathbb{R}^k \mid \neg\neg(\exists c \in K.\ p \in c)\}$$

Lemma 6. $|K|$ *is compact.*

Proof. Since $\{p : \mathbb{R}^k \mid p \in c\}$ is a compact subset of $\mathbb{R}^k$, this lemma follows from Theorem 2. □

Lemma 7. $|K'| = |K|$.

Proof. If $c \in K$ and h is a half cube of c, and $p \in h$ then $p \in c$, so $|K'| \subseteq |K|$. If $p \in c$ for some $c \in K$ then for each $i < k$, $\neg\neg((p_i \leq m_i) \vee (m_i \leq p_i))$ where m_i is the midpoint of the i^{th} interval of c. This lets us prove

$$\neg\neg(\exists h.\ p \in h \wedge h \text{ half of } c)$$

So, $|K| \subseteq |K'|$. □

Note that Lemmas 6 and 7 would not be provable had we defined the polyhedron of a complex to be the normal union of its cubes rather than the stable union of its cubes.

5 No-retraction theorem

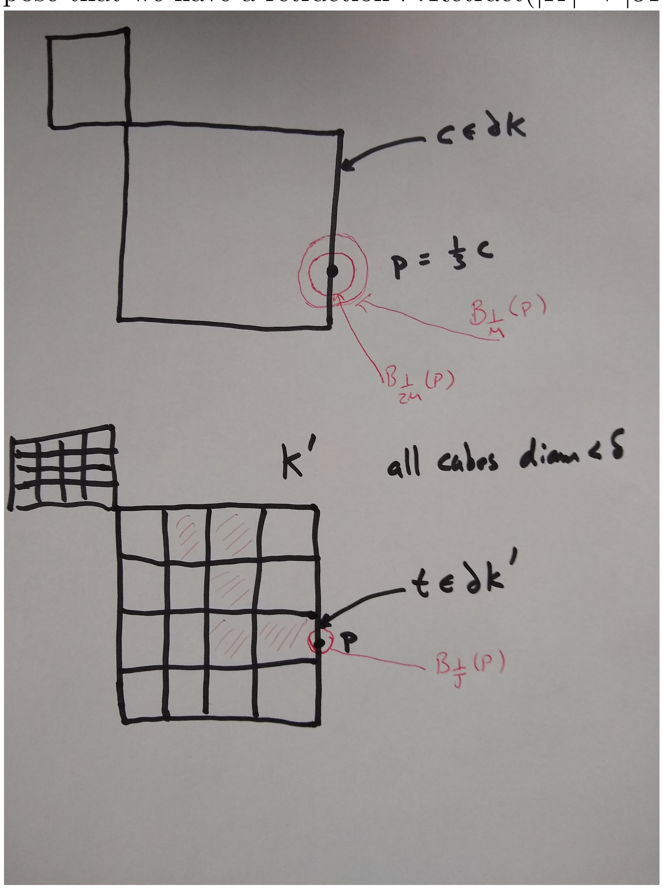
A *boundary retraction* for K is a function $r \in \text{FUN}(|K| \to |\partial K|)$ such that for all $x \in |\partial K|$, $r(x) \equiv x$. We write $\text{Retract}(|K| \to |\partial K|)$ for the type of boundary retractions for K.

Theorem 4. *(No-retraction) For every $n \geq 0$, for every n-dimensional complex K,*

$$length(K) > 0 \Rightarrow \neg Retract(|K| \to |\partial K|)$$

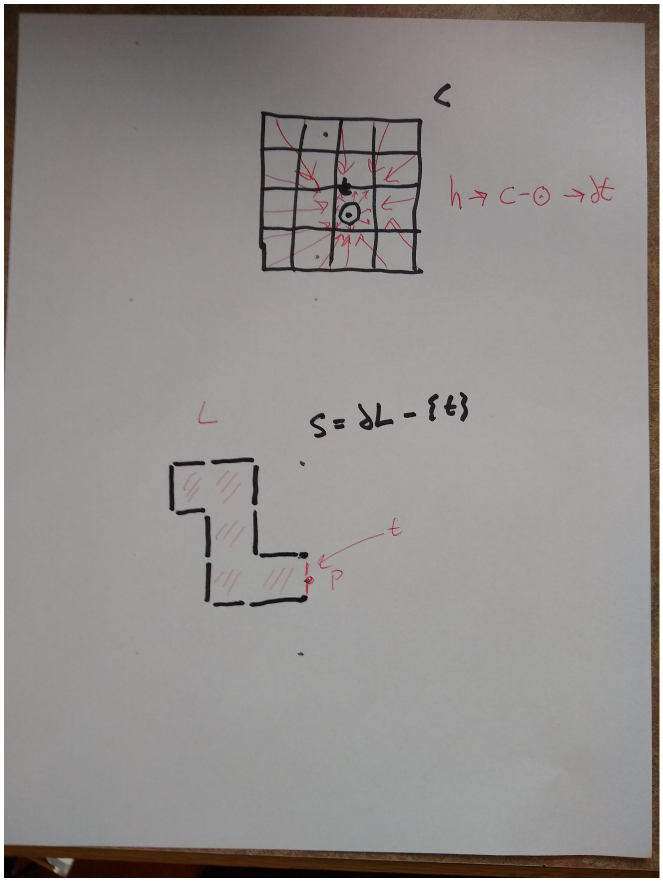
Proof. We prove the theorem by induction on the dimension n. When $n = 0$ the theorem is easy because $|K| \neq \emptyset$ but $|\partial K| = \emptyset$. So there can't be any map from $|K|$ to $|\partial K|$.

We now have $n > 0$ and the induction hypothesis that there are no boundary retractions for non-empty complexes of dimension $n - 1$. We suppose that we have a retraction r: $\text{Retract}(|K| \to |\partial K|)$ and prove False.



We enumerate the steps in the proof and justify them afterward.

1. Choose $c \in \partial K$.
2. Choose $p \in c$ at the 1/3 point of each dimension.
3. Choose M so that $d(p, |\partial K \setminus \{c\}|) > 1/M$.
4. Find $\delta > 0$ so that $d(x, y) < \delta \Rightarrow d(r(x), r(y)) < 1/2M$
5. Subdivide K enough times that every cube in K' is smaller than δ.
6. Find $t \in \partial K'$ with $p \in t$. (Then $|t| \subseteq |c|$ and p is at 1/3 or 2/3 of t).
7. Find $J \geq 2M$ so that $d(p, |K' \setminus \{t\}|) > 1/J$
8. Let $RN(e) \Leftrightarrow \exists x \in e.\ d(r(x), p) < 1/J$.
9. Let $L = \{e \in K' \mid RN(e)\}$.
10. Note $r(|L|) \subseteq |c|$.
11. Note $RN(f) \Rightarrow (f \in \partial L \Leftrightarrow f \in \partial K')$.
12. Let $S = \partial L \setminus \{t\}$. Note $\dim(S) = n - 1$ and $\text{length}(S) > 0$.
13. Note $d(p, r(|S|)) \geq 1/J$.
14. $\partial S = (\partial(\partial L) \bigtriangleup \partial\{t\}) = \partial\{t\} = \text{faces of } t$
15. $|S| \subset |L|$, so $r(|S|) \subset |c| \setminus \{x \mid d(x, p) < 1/J\}$
16. There is a retraction h from $|c| \setminus \{x \mid d(x, p) < 1/J\}$ to |faces of t|.
17. Then $h \circ r$ is a boundary retraction for S, which contradicts the induction hypothesis.



Justifications:

1. Since we have a retraction, ∂K must be non null.

2. By the 1/3 point we simply mean that $p_i = \frac{2a_i+b_i}{3}$ where $[a_i, b_i]$ is the i^{th} interval of c. Sieclucki invokes the Baire category theorem at this point, but that is not needed. The reason to use 1/3 is that when we subdivide K finitely many times to get K' by dividing the intervals of its cubes by two, the 1/3 point will not end up in the boundary of any cube in K'.

3. Since we are proving False, we use cases on $d(p, |\partial K \setminus \{c\}|) > 0$. If it were false then the distance would be 0 and $p \in |\partial K \setminus \{c\}|$. But by the compatibility of the cubes in ∂K no other cube in ∂K can contain p because p is not in a proper face of c.

4. By Theorem 1 and Lemma 6, r is uniformly continuous.

5. Subdivision decreases the diameter of the cubes by a factor of 1/2, so for some $j : \mathbb{N}$ the maximum diameter of the iterated subdivision $K^{('j)}$ is less than δ. We use K' in the rest of the proof as short for $K^{('j)}$.

6. Since $p \in c$, and $c \in \partial K$, we can use the argument in Lemma 7 to prove $\neg\neg(\exists h.\ p \in h \wedge h$ half of $c)$. Iterating this j times, and using Lemma 3, we have

$$\neg\neg(\exists t \in \partial K'.\ p \in t \wedge t \text{ iterated half of } c)$$

Since we are proving False we can strengthen this to $\exists t \in K'.\ p \in t \wedge (t$ iterated half of $c))$. By induction on j we can then prove that the coordinates of p are at either the 1/3 or 2/3 points of the intervals of t.

7. Point p is not in a proper face of t so we can use the same reasoning we used for step 3.

8. The predicate $RN(e)$ is to be read “retracts near“. It holds when cube e contains a point that retracts near (within $1/J$) to p .

9. We need L to be a *list* (a sub-list of K'). Since $RN(e)$ is undecidable, it would not be possible to find L in general but since we are proving False, it is. From $\forall e \in K'.\ \neg\neg(RN(e) \vee \neg RN(e))$ we get, by induction on the length of K', $\neg\neg(\forall e \in K'.\ RN(e) \vee \neg RN(e))$. This move is call the *finite double negation shift*. Since we are proving False, we strengthen to $\forall e \in K'.\ RN(e) \vee \neg RN(e)$ and use this to decide which cubes e are in the list L.

10. If $z \in |L|$ then we prove $r(z) \in c$. This is stable so we can get an $e \in K'$ with $RN(e)$ and $z \in e$. By $RN(e)$ there is an $x \in e$ with $d(r(x), p) < 1/J \leq 1/2M$. By steps 4 and 5, $d(r(z), r(x)) < 1/2M$ so $d(r(z), p) < 1/M$. So by step 3, we must have $r(z) \in c$.

11. $RN(f) \Rightarrow (f \in \partial L \Leftrightarrow f \in \partial K')$ this is the key observation in the whole argument. If f contains a point x that retracts near p, then if $f \leq e$ then e also contains x so $RN(e)$. Thus, $f \leq e \Rightarrow (e \in L \Leftrightarrow e \in K')$. Therefore the number of cubes in L of which f is a face is the same as the number of cubes in K' of which f is a face (because they are the same set of cubes). Thus one is odd if and only if the other is odd (since they are the same number).

12. ∂L is an $n-1$-dimensional complex. Since $RN(t)$ and $t \in \partial K'$ the previous step shows that $t \in \partial L$. By Lemma 5, length(∂L) is even, so it must be at least two, since $t \in \partial L$. Thus, when we remove t from ∂L to get S, length(S) > 0.

13. Suppose $x \in |S|$. We must prove $d(p, r(x)) \geq 1/J$, which is a stable proposition so it is enough to prove $\neg(d(p, r(x)) < 1/J)$. Then for some $f \in \partial L \setminus \{t\}$, $x \in f$. If $d(p, r(x)) < 1/J$ then $RN(f)$ so by step 11, $f \in \partial K'$. By Lemmas 7 and 3, $|\partial K'| = |\partial K|$ so since r is a retraction, $r(x) \equiv x$ so we have $d(p, x) < 1/J$ contradicting step 7.

14. This follows from Lemmas 4 and 2.

15. This follows from steps 10, 12, and 13.

16. We have $t \subseteq c$, $p \in t$ and p is not in any proper face of t. We construct a retraction from $c^{-} = \{x : \mathbb{R}^k \mid x \in c \ \wedge \ x\#p\}$ to |faces of t|. For any $i < k$ where the i^{th} interval of c is a 0-dimensional $[a_i, a_i]$ we have that all points $x \in c$ have $x_i \equiv a_i$, so we can use a homeomorphism that carries c, c^{-}, t, and p to corresponding objects in $\mathbb{R}^m$ where $m = n - 1 = \dim(c)$. Hence, we can assume that the intervals $[a_i, b_i], i < m$ for t are proper, $a_i < b_i$, and that $\forall i < m.\ a_i < p_i < b_i$. Now |faces of t| $= \{x : \mathbb{R}^m \mid x \in t \ \wedge \ \exists j < m.(x_j \equiv a_j) \vee (x_j \equiv b_j)\}$. Now we can use a homeomorphism that carries the box t to the unit cube $\{x : \mathbb{R}^m \mid \|x\|_\infty \leq 1\}$ and then use the homeomorphism that carries that unit cube to the unit Euclidean ball $B(m)$. The point p in the interior of t is carried to a point in the interior of the ball $B(m)$. Finally, we construct the retraction from *punctured* $\mathbb{R}^m$ to the boundary of $B(m)$ as we did in the reduction of Brouwer's fixedpoint theorem to the no-retraction theorem, namely by intersecting a ray with a sphere using the quadratic formula.

    This step of Sieclucki's argument, which is just one sentence in his paper, was perhaps the most tedious bit of the formal Nuprl proof, but we now have many lemmas about homeomorphisms and retractions in our Nuprl library for possible future use.

17. This follows from steps 14 and 16.

□

Remarks. Sieclucki's original proof at step 2 said that by the Baire category theorem there exists a point p that was in the *interior* of c (for Sieclucki, c was a simplex s) and p was in the interior of any simplex containing p in any of the barycentric subdivisions $K', K'', \dots K^{('j)} \dots$. When $n = 1$ the dimension of c is 0 so it is a single point, and under the normal definition of interior, the interior of c would be empty. Thus, we were convinced that Sieclucki's proof was incorrect for the case $n = 1$. We made a separate argument for the case $n = 1$, and proved the general case only for $n > 1$. But then we found that the formal proof for the general case did not rely on the assumption $n > 1$ except in one place, in step 12 where we prove that length$(S) > 0$ (which is something that Sieclucki neglects to prove). We had proved that using step 14, and the fact that $\partial\{t\} \neq \emptyset$ when $n > 1$ (because then $\dim(t) > 0$). To get the whole argument to work for $n > 0$ we instead used our Lemma 5 which is true for cubes but not for simplexes.

6 Brouwer's fixedpoint theorem

From the no-retraction theorem for complexes K we get the corollary that for a single rational cube c there is no retraction from $\{x : \mathbb{R}^k \mid x \in c\}$ to $|\text{faces of } c|$. From this we can get that there is no-retraction from the unit ball $B(k) = \{x : \mathbb{R}^k \mid |||x||| \leq 1\}$ to its boundary $\{x : \mathbb{R}^k \mid |||x||| = 1\}$ by using the homeomorphism that takes the unit rational cube to the unit ball.

As discussed in Section 3, we have now proved Theorem 3, Brouwer's fixedpoint theorem.