Problem Set 3 Due Thursday October 14 CS4860 Fall 2010

Reading: Please read Smullyan Chapters IV and V.

1 Problem 1

Study all of the exercises on page 56 and write tableau proofs for the following, some of which are not in the textbook.

1. $(\forall x)(Px \rightarrow (\exists x)Px)$
2. $(\forall x)[Px \rightarrow C] \Leftrightarrow [(\exists x)Px \rightarrow C]$
3. $\sim (\exists x)Px \rightarrow (\forall x) \sim Px$
4. $\sim (\forall x)Px \rightarrow (\exists x) \sim Px$

2 Problem 2

Solve the exercise $(H \wedge K) \rightarrow L$ at the top of page 57.

3 Problem 3

Solve the exercise at the bottom of page 63.

4 Problem 4

A *Monoid* is an algebraic structure whose domain of discourse (universe) we call M; it has one associative *binary operator* $\times$ and a *unit* element 1. Below are the two axioms for Monoids. Using the predicate $Unit(x)$ to assert that x is a unit element and the predicate $Op(z, x, y)$ to mean $z = x \times y$ and $Eq(x, y)$ to mean x = y in M.

(**a**) write the two axioms for a Monoid in First-Order Logic and
(**b**) prove that the unit element is unique, i.e. any two units are equal; that is, prove in FOL that the two axioms imply uniqueness of unit. First write your proof informally and then give a tableau proof.

Axiom 1: $(x \times y) \times z = x \times (y \times z)$.
Axiom 2: $1 \times x = x \times 1 = x$ where is the unit.

5 Problem 5

Using the predicates $Boole(x)$, $True(x)$, $False(x)$, $Eq(x, y)$, $Bnot(x, y)$, and $Bor(z, x, y)$ write First-Order Logic formulas that define the truth table for the *not* and *or* operators. We intend $True(x)$ to mean that x is the element t, $Bnot(x, y)$ to mean $x =\sim y$ and $Bor(z, x, y)$ to mean that z is the *or* of x and y, $z = x \vee y$.

6 Problem 6

Using the predicates listed below, state in First-Order Logic (FOL) the following Boolean valuation theorem of Smullyan page 10-11:
"*For any formula X and any valuation v_0 of the propositional variables of X, there is one (and only one) valuation v of all subformulas of X which extends v_0 and is a Boolean valuation on all subformulas of X.*" Leave out the "and only one" in your FOL formulation.

Formula(x) – x is a propositional formula
Subform(y,x) – y is a subformula of x
Valuation(v) – v is a valuation
Value(v,x,b) – the value of v on x is the Boolean b (sensible only for v a Valuation)
Boolean(v,x) – v is a Boolean valuation of the subformulas of x. (Smullyan's Def 1 on page 10 can be read as the definition of this predicate where we replace E by the formula x.)

7 Problem 7

Translate the following expressions into English.

Let $Form \rightarrow \mathbb{B}$ be the type of *valuations*. Let $Set(Form)$ be the type of sets of formulas, and let $Var(S)$ be the type of variables in the set S. Let $\mathbb{N}_n^+$ be the set of numbers from 1 to n, i.e. $\{1, ..., n\}$. We use two kinds of parentheses, (...) and [...] for readability of the formulas.

1. $\forall S : Set(Form)).[\forall v : Var(S) \rightarrow \mathbb{B}.\ \exists X : Form.(X \in S \,\&\, bval(X, v) = t)] \Rightarrow \exists n : \mathbb{N}.\ \exists f : \mathbb{N}_n^+ \rightarrow S.\ TAUT(f(1) \vee ... \vee f(n))$.

2. $\forall f : \mathbb{N} \rightarrow Form.(\ [\forall v : Var \rightarrow \mathbb{B}.\ \exists k : \mathbb{N}.\ bval(f((k)), v) = t] \Rightarrow \exists n : \mathbb{N}.\ \exists g : \mathbb{N}_n^+ \rightarrow \mathbb{N}.\ TAUT(f(g(1)) \vee ... \vee f(g(n)))\ )$.

3. Are these two statements equivalent in the sense that they imply each other? Explain your answer.
4. Is either of these statements true? Explain.

8 Extra Credit

We can state our Strong Boolean Evaluation Theorem about bval as folows: *There is an evaluation f such that for any formula x and any interpretation i of the variables of x, there is a valuation v that extends i and is a Boolean valuation of the subformulas of x.*

(**a**) Use the predicates listed below and in Problem 6 to formalize this theorem in FOL.

(**b**) It will not be possible to conduct any proofs without axioms for these predicates. Write an axiom for Eval relating it to *v*.

Evaluation(f) – f is an evaluation (think of the relation bval(x,i) = b)
Interpretation(i,x) – i is an assignment of Booleans to the variables of x
Value(i,x,y,b) – b is the value of the interpretation i applied to the variable y of formula x
Eval(f,x,i,b) – f applied to formula x under interpretation i gives Boolean b.