CHAPTER 2

PROPOSITIONAL INTUITIONISTIC LOGIC

PROOF THEORY

§ 1. Beth tableaus

In this section we present a modified version of a proof system due originally to Beth. It is based on [2, § 145], but at the suggestion of R. Smullyan, we have introduced signed formulas and single trees in place of the unsigned formulas and dual trees of Beth.

By a *signed formula* we mean TX or FX where X is a formula. If S is a set of signed formulas and H is a single signed formula, we will write $S \cup \{H\}$ simply as $\{S, H\}$ or sometimes S, H.

First we state the *reduction rules*, then we describe their use; S is any set (possibly empty) of signed formulas, and X and Y are any formulas:

$$T\wedge \quad \frac{S, T(X \wedge Y)}{S, TX, TY} \qquad\qquad F\wedge \quad \frac{S, F(X \wedge Y)}{S, FX \mid S, FY}$$

$$T\vee \quad \frac{S, T(X \vee Y)}{S, TX \mid S, TY} \qquad\qquad F\vee \quad \frac{S, F(X \vee Y)}{S, FX, FY}$$

$$T\sim \quad \frac{S, T(\sim X)}{S, FX} \qquad\qquad F\sim \quad \frac{S, F(\sim X)}{S_T, TX}$$

$$T\supset \quad \frac{S, T(X \supset Y)}{S, FX \mid S, TY} \qquad\qquad F\supset \quad \frac{S, F(X \supset Y)}{S_T, TX, FY}$$

In rules $F\sim$ and $F\supset$ above, S_T means $\{TX \mid TX \in S\}$.

$$\frac{FA}{T\sim A \mid TA}$$

Remark 1.1: S is a set, and hence $\{S, TX\}$ is the same as $\{S, TX, TX\}$. Thus duplication and elimination rules are not necessary.

If U is a set of signed formulas, we say one of the above rules, call it rule R, *applies to* U if by appropriate choice of S, X and Y the collection of signed formulas above the line in rule R becomes U.

By an *application of rule* R *to the set* U we mean the replacement of U by U_1 (or by U_1 and U_2 if R is $F\wedge$, $T\vee$ or $T\supset$) where U is the set of formulas above the line in rule R (after suitable substitution for S, X and Y) and U_1 (or U_1, U_2) is the set of formulas below. This assumes R applies to U. Otherwise the result is again U. For example, by applying rule $F\supset$ to the set $\{TX, FY, F(Z\supset W)\}$ we may get the set $\{TX, TZ, FW\}$. By applying rule $T\vee$ to the set $\{TX, FY, T(Z\vee W)\}$ we may get the two sets $\{TX, FY, TZ\}$ and $\{TX, FY, TW\}$.

By a *configuration* we mean a finite collection $\{S_1, S_2, \ldots, S_n\}$ of sets of signed formulas.

By an *application of the rule* R *to the configuration* $\{S_1, S_2, \ldots, S_n\}$ we mean the replacement of this configuration with a new one which is like the first except for containing instead of some S_i the result (or results) of applying rule R to S_i.

By a *tableau* we mean a finite sequence of configurations $\mathscr{C}_1, \mathscr{C}_2, \ldots, \mathscr{C}_n$ in which each configuration except the first is the result of applying one of the above rules to the preceding configuration.

A set S of signed formulas is *closed* if it contains both TX and FX for some formula X. A configuration $\{S_1, S_2, \ldots, S_n\}$ is closed if each S_i in it is closed. A tableau $\mathscr{C}_1, \mathscr{C}_2, \ldots, \mathscr{C}_n$ is closed if some $\mathscr{C}_i$ in it is closed.

By a *tableau for a set S of signed formulas* we mean a tableau $\mathscr{C}_1, \mathscr{C}_2, \ldots, \mathscr{C}_n$ in which $\mathscr{C}_1$ is $\{S\}$. A finite set of signed formulas S is *inconsistent* if some tableau for S is closed. Otherwise S is *consistent*. X is a *theorem* if $\{FX\}$ is inconsistent, and a closed tableau for $\{FX\}$ is called a *proof of X*. If X is a theorem we write $\vdash_I X$.

We will show in the next few sections the correctness and completeness of the above system relative to the semantics of ch. 1.

Examples of proofs in this system may be found in § 5.

The corresponding classical tableau system is like the above, but in rules $F\sim$ and $F\supset$, S_T is replaced by S (see [20]). The interpretations of the classical and intuitionistic systems are different.

In the classical system TX and FX mean X is true and X is false :spectively. The rules may be read: if the situation above the line is the ase, the situation below the line is also (or one of them is, if the rule is isjunctive: $F\wedge$, $T\vee$, $T\supset$). Thus TX means the same as X, and FX ıeans $\sim X$. Classically the signs T and F are dispensable. Proof is a :futation procedure. Suppose X is not true (begin a tableau with FX). :onclude that some formula must be both true and not true (a closed onfiguration is reached). Since this can not happen, X is true.

In the intuitionistic case TX is to mean X is known to be true (X is roven). FX is to mean X is not known to be true (X has not been roved). The rules are to be read: if the situation above the line is the ıse, then the situation below the line is possible, i.e. compatible with ur present knowledge (if the rule is disjunctive, one of the situations elow the line must be possible). For example consider rule $F\supset$. If we ave not proved $X\supset Y$, it is possible to prove X without proving Y, for 'this were not possible, a proof of Y would be 'inherent' in a proof of X, nd this fact would constitute a proof of $X\supset Y$. But we have S_T below ıe line in this rule and not S because in proving X we might inadvertently erify some additional previously unproven formula (some $FZ\in S$ might ecome TZ). Similarly for $F\sim$. The proof procedure is again by refuta- on. Suppose X is not proven (begin a tableau with FX). Conclude that is possible that some formula is both proven and not proven. Since this ; impossible, X is proven.

We have presented this system in a very formal fashion because it ıakes talking about it easier. In practice there are many simplifications hich will become obvious in any attempt to use the method. Also, roofs may be written in a tree form. We find the resulting simplified ystem the easiest to use of all the intuitionistic proof systems, except in ɔme cases, the system resulting by the same simplifications from the losely related one presented in ch. 6 § 4. A full treatment of the corre- ponding classical tableau system, with practical simplifications, may be ɔund in [20].

2. Correctness of Beth tableaus

)efinition 2.1: We call a set of signed formulas

$$\{TX_1, ..., TX_n, FY_1, ..., FY_m\}$$

realizable if there is some model $\langle \mathscr{G}, \mathscr{R}, \vDash \rangle$ and some $\Gamma\in\mathscr{G}$ such that $\Gamma\vDash X_1, ..., \Gamma\vDash X_n$, $\Gamma\nvDash Y_1, ..., \Gamma\nvDash Y_m$. We say that Γ *realizes* the set.

If $\{S_1, S_2, ..., S_n\}$ is a configuration, we call it realizable if some S_i in it is realizable.

Theorem 2.2: Let $\mathscr{C}_1, \mathscr{C}_2, ..., \mathscr{C}_n$ be a tableau. If $\mathscr{C}_i$ is realizable, so is $\mathscr{C}_{i+1}$.

Proof: We have eight cases, depending on the rule whose application produced $\mathscr{C}_{i+1}$ from $\mathscr{C}_i$.

Case (1): $\mathscr{C}_i$ is $\{..., \{S, T(X\vee Y)\}, ...\}$ and $\mathscr{C}_{i+1}$ is $\{..., \{S, TX\}, \{S, TY\}, ...\}$. Since $\mathscr{C}_i$ is realizable, some element of it is realizable. If that element is not $\{S, T(X\vee Y)\}$, the same element of $\mathscr{C}_{i+1}$ is realizable. If that element is $\{S, T(X\vee Y)\}$, then for some model $\langle \mathscr{G}, \mathscr{R}, \vDash \rangle$ and some $\Gamma\in\mathscr{G}$, Γ realizes $\{S, T(X\vee Y)\}$. That is, Γ realizes S and $\Gamma\vDash(X\vee Y)$. Then $\Gamma\vDash X$ or $\Gamma\vDash Y$, so either Γ realizes $\{S, TX\}$ or $\{S, TY\}$. In either case $\mathscr{C}_{i+1}$ is realizable.

Case (2): C_i is $\{..., \{S, F(\sim X)\}, ...\}$ and $\mathscr{C}_{i+1}$ is $\{..., \{S_T, TX\}, ...\}$. $\mathscr{C}_i$ is realizable, and it suffices to consider the case that $\{S, F(\sim X)\}$ is the realizable element. Then there is a model $\langle \mathscr{G}, \mathscr{R}, \vDash \rangle$ and a $\Gamma\in\mathscr{G}$ such that Γ realizes S and $\Gamma\nvDash\sim X$. Since $\Gamma\nvDash\sim X$, for some $\Gamma^*\in\mathscr{G}$, $\Gamma^*\vDash X$. But clearly, if Γ realizes S, Γ^* realizes S_T (by theorem 1.4.4). Hence Γ^* realizes $\{S_T, TX\}$ and $\mathscr{C}_{i+1}$ is realizable.

The other six cases are similar.

Corollary 2.3: The system of Beth tableaus is correct, that is, if $\vdash_I X$, X is valid.

Proof: We show the contrapositive. Suppose X is not valid. Then there is a model $\langle \mathscr{G}, \mathscr{R}, \vDash \rangle$ and a $\Gamma\in\mathscr{G}$ such that $\Gamma\nvDash X$. In other words $\{FX\}$ is realizable. But a proof of X would be a closed tableau $\mathscr{C}_1, \mathscr{C}_2, ..., \mathscr{C}_n$ in which $\mathscr{C}_1$ is $\{\{FX\}\}$. But $\mathscr{C}_1$ is realizable, hence each $\mathscr{C}_i$ is realizable. But obviously a realizable configuration cannot be closed. Hence $\nvdash_I X$.

§ 3. Hintikka collections

In classical logic a set S of signed formulas is sometimes called downward saturated, or a Hintikka set, if

$$TX\wedge Y\in S \Rightarrow TX\in S \text{ and } TY\in S,$$
$$FX\vee Y\in S \Rightarrow FX\in S \text{ and } FY\in S,$$

$$
\begin{array}{lll}
TX \vee Y \in S & \Rightarrow & TX \in S \quad \text{or} \quad TY \in S, \\
FX \wedge Y \in S & \Rightarrow & FX \in S \quad \text{or} \quad FY \in S, \\
T \sim X \in S & \Rightarrow & FX \in S, \\
TX \supset Y \in S & \Rightarrow & FX \in S \quad \text{or} \quad TY \in S, \\
F \sim X \in S & \Rightarrow & TX \in S, \\
FX \supset Y \in S & \Rightarrow & TX \in S \quad \text{and} \quad FY \in S.
\end{array}
$$

Remark 3.1: The names Hintikka set and downward saturated set were given by Smullyan [20]. Hintikka, their originator, called them model sets.

Hintikka showed that any consistent downward saturated set could be included in a set for which the above properties hold with $\Rightarrow$ replaced by $\Leftrightarrow$. From this follows the completeness of certain classical tableau systems. This approach is thoroughly developed by Smullyan in [20].

We now introduce a corresponding notion in intuitionistic logic, which we call a Hintikka collection. While its intuitive appeal may not be as immediate as in the classical case, its usefulness is as great.

Definition 3.2: Let $\mathscr{G}$ be a collection of consistent sets of signed formulas. We call $\mathscr{G}$ a *Hintikka collection* if for any $\Gamma \in \mathscr{G}$

$$
\begin{array}{lll}
TX \wedge Y \in \Gamma & \Rightarrow & TX \in \Gamma \quad \text{and} \quad TY \in \Gamma, \\
FX \vee Y \in \Gamma & \Rightarrow & FX \in \Gamma \quad \text{and} \quad FY \in \Gamma, \\
TX \vee Y \in \Gamma & \Rightarrow & TX \in \Gamma \quad \text{or} \quad TY \in \Gamma, \\
FX \wedge Y \in \Gamma & \Rightarrow & FX \in \Gamma \quad \text{or} \quad FY \in \Gamma, \\
T \sim X \in \Gamma & \Rightarrow & FX \in \Gamma, \\
TX \supset Y \in \Gamma & \Rightarrow & FX \in \Gamma \quad \text{or} \quad TY \in \Gamma, \\
F \sim X \in \Gamma & \Rightarrow & \text{for some} \quad \Delta \in \mathscr{G}, \Gamma_T \subseteq \Delta \quad \text{and} \quad TX \in \Delta, \\
FX \supset Y \in \Gamma & \Rightarrow & \text{for some} \quad \Delta \in \mathscr{G}, \Gamma_T \subseteq \Delta, TX \in \Delta, FY \in \Delta.
\end{array}
$$

Definition 3.3: Let $\mathscr{G}$ be a Hintikka collection. We call $\langle \mathscr{G}, \mathscr{R}, \vDash \rangle$ a *model for* $\mathscr{G}$ if

(1). $\langle \mathscr{G}, \mathscr{R}, \vDash \rangle$ is a model,

(2). $\Gamma_T \subseteq \Delta \Rightarrow \Gamma \mathscr{R} \Delta$,

(3). $TX \in \Gamma \Rightarrow \Gamma \vDash X$,
$FX \in \Gamma \Rightarrow \Gamma \nvDash X$.

Theorem 3.4: There is a model for any Hintikka collection.

Proof: Let $\mathscr{G}$ be a Hintikka collection. Define $\mathscr{R}$ by: $\Gamma \mathscr{R} \Delta$ if $\Gamma_T \subseteq \Delta$.

If A is atomic, let $\Gamma \vDash A$ if $TA \in \Gamma$, and extend $\vDash$ to produce a model $\langle \mathscr{G}, \mathscr{R}, \vDash \rangle$. To show property (3) is a straightforward induction on the degree of X. We give one case as illustration. Suppose X is $\sim Y$ and the result is known for Y. Then

$$
\begin{array}{lll}
T \sim Y \in \Gamma & \Rightarrow & (\forall \Delta \in \mathscr{G})(\Gamma_T \subseteq \Delta \Rightarrow T \sim Y \in \Delta) \\
 & \Rightarrow & (\forall \Delta \in \mathscr{G})(\Gamma_T \subseteq \Delta \Rightarrow FY \in \Delta) \\
 & \Rightarrow & (\forall \Delta \in \mathscr{G})(\Gamma \mathscr{R} \Delta \Rightarrow \Delta \nvDash Y) \\
 & \Rightarrow & \Gamma \vDash \sim Y,
\end{array}
$$

and

$$
\begin{array}{lll}
F \sim Y \in \Gamma & \Rightarrow & (\exists \Delta \in \mathscr{G})(\Gamma_T \subseteq \Delta \text{ and } TY \in \Delta) \\
 & \Rightarrow & (\exists \Delta \in \mathscr{G})(\Gamma \mathscr{R} \Delta \quad \text{and } \Delta \vDash Y) \\
 & \Rightarrow & \Gamma \nvDash \sim Y.
\end{array}
$$

It follows from this theorem that to show the completeness of Beth tableaus we need only show the following: If $\nvdash_I X$, then there is a Hintikka collection $\mathscr{G}$ such that for some $\Gamma \in \mathscr{G}$, $FX \in \Gamma$.

§ 4. Completeness of Beth tableaus

Let S be a set of signed formulas. By $\mathscr{S}(S)$ we mean the collection of all signed subformulas of formulas in S. If S is finite, $\mathscr{S}(S)$ is finite.

Let S be a finite, consistent set of signed formulas. We define a *reduced set for S* (there may be many) as follows:

Let S_0 be S. Having defined S_n, a finite consistent set of signed formulas, suppose one of the following Beth reduction rules applies to S_n: $T\wedge$, $F\wedge$, $T\vee$, $F\vee$, $T\sim$ or $T\supset$. Choose one which applies, say $F\wedge$. Then S_n is $\{U, FX \wedge Y\}$. This is consistent, so clearly either $\{U, FX \wedge Y, FX\}$ or $\{U, FX \wedge Y, FY\}$ is consistent. Let S_{n+1} be $\{U, FX \wedge Y, FX\}$ if consistent, otherwise let S_{n+1} be $\{U, FX \wedge Y, FY\}$. Similarly if $T\wedge$ applies and was chosen, then S_n is $\{U, TX \wedge Y\}$. Since this is consistent, $\{U, TX \wedge Y, TX, TY\}$ is consistent. Let this be S_{n+1}. In this way we define a sequence $S_0, S_1, S_2, \ldots$. This sequence has the property $S_n \subseteq S_{n+1}$. Further, each S_n is finite and consistent. Since each $S_n \subseteq \mathscr{S}(S)$, there are only a finite number of different possible S_n. Consequently there must be a member of the sequence, say S_n, such that the application of any one of the rules (except $F\sim$ or $F\supset$) produces S_n again. Call such an S_n a *reduced* set of S, and denote it by S'. Clearly any finite, consistent set of signed formulas has a finite, consistent reduced set. Moreover, if S' is a reduced set, it has the following suggestive properties:

$$
\begin{array}{lll}
TX \wedge Y \in S' & \Rightarrow TX \in S' \text{ and } & TY \in S', \\
FX \vee Y \in S' & \Rightarrow FX \in S' \text{ and } & FY \in S', \\
TX \vee Y \in S' & \Rightarrow TX \in S' \text{ or } & TY \in S', \\
FX \wedge Y \in S' & \Rightarrow FX \in S' \text{ or } & FY \in S', \\
T \sim X \in S' & \Rightarrow FX \in S', & \\
TX \supset Y \in S' & \Rightarrow FX \in S' \text{ or } & TY \in S', \\
S' \text{ is consistent.} & &
\end{array}
$$

Now, given any finite, consistent set of signed formulas S, we form the collection of *associated sets* as follows:

If $F{\sim}X \in S$, $\{S_T, TX\}$ is an associated set.

If $FX \supset Y \in S$, $\{S_T, TX, FY\}$ is an associated set.

Let $\mathscr{A}(S)$ be the collection of all associated sets of S. $\mathscr{A}(S)$ is finite, since $U \in \mathscr{A}(S)$ implies $U \subseteq \mathscr{S}(S)$ and $\mathscr{S}(S)$ is finite. $\mathscr{A}(S)$ has the following properties: if S is consistent, any associated set is consistent and

$$
\begin{array}{ll}
F \sim X \in S & \Rightarrow \text{ for some } U \in \mathscr{A}(S)\ \ S_T \subseteq U,\ \ TX \in U, \\
FX \supset Y \in S & \Rightarrow \text{ for some } U \in \mathscr{A}(S)\ \ S_T \subseteq U,\ \ TX \in U,\ \ FY \in U.
\end{array}
$$

Now we proceed with the proof of completeness.

Suppose $\nvdash_I X$. Then $\{FX\}$ is consistent. Extend it to its reduced set S_0. Form $\mathscr{A}(S_0)$. Let the elements of $\mathscr{A}(S_0)$ be $U_1, U_2, \ldots, U_n$. Let S_1 be the reduced set of $U_1, \ldots, S_n$ be the reduced set of U_n. Thus, we have the sequence $S_0, S_1, S_2, \ldots, S_n$.

Next form $\mathscr{A}(S_1)$. Call its elements $U_{n+1}, U_{n+2}, \ldots, U_m$. Let S_{n+1} be the reduced set of U_{n+1} and so on. Thus, we have the sequence $S_0, S_1, \ldots, S_n, S_{n+1}, \ldots, S_m$. Now we repeat the process with S_2, and so on.

In this way we form a sequence $S_0, S_1, S_2, \ldots$. Since each $S_i \subseteq \mathscr{S}(S)$, there are only finitely many possible different S_i. Thus we must reach a point S_k of the sequence such that any continuation repeats on earlier member.

Let $\mathscr{G}$ be the collection $\{S_0, S_1, \ldots, S_k\}$. It is easy to see that $\mathscr{G}$ is a Hintikka collection. But $FX \in S_0 \in \mathscr{G}$. Thus we have shown:

Theorem 4.1: Beth tableaus are complete.

Remark 4.2: This proof also establishes that propositional intuitionistic logic is decidable. For, if we follow the above procedure beginning with FX, after a finite number of steps we will have either a closed tableau for $\{FX\}$ or a counter-model for X. Moreover, the number of steps may be bounded in terms of the degree of X.

The completeness proof presented here is in essence the original proof of Kripke [13]. For a different tableau completeness proof see ch. 5 § 6, where it is given for first order logic. For a completeness proof of an axiom system see ch. 5 § 10, where it also is given for a first order system. The work in ch. 1 § 6 provides an algebraic completeness proof, since the Lindenbaum algebra of intuitionistic logic is easily shown to be a pseudo-boolean algebra. See [16].

§ 5. Examples

In this section, so that the reader may gain familiarity with the foregoing, we present a few theorems and non-theorems of intuitionistic propositional logic, together with their proofs or counter-models.

We show

(1). $\nvdash_I A \vee \sim A$,

(2). $\vdash_I \sim\sim(A \vee \sim A)$,

(3). $\nvdash_I \sim\sim A \supset A$,

(4). $\vdash_I (A \vee B) \supset \sim(\sim A \wedge \sim B)$,

(5). $\nvdash_I \sim\sim(A \vee B) \supset (\sim\sim A \vee \sim\sim B)$.

For the general principle connecting (1) and (2) see ch. 4 § 8.

(1). $\nvdash_I A \vee \sim A$.

A counter example for this is the following:

$$
\mathscr{G} = \{\Gamma, \Delta\}
$$
$$
\Gamma \mathscr{R} \Gamma, \quad \Gamma \mathscr{R} \Delta, \quad \Delta \mathscr{R} \Delta.
$$

$\Delta \vDash A$ is the $\vDash$ relation for atomic formulas, and $\vDash$ is extended to all formulas as usual. We may schematically represent this model by

$$
\begin{array}{c}
\Gamma \\
| \\
\Delta \vDash A
\end{array}
$$

We claim $\Gamma \nvDash A \vee \sim A$. Suppose not. If $\Gamma \vDash A \vee \sim A$, either $\Gamma \vDash A$ or $\Gamma \vDash \sim A$. But $\Gamma \nvDash A$. If $\Gamma \vDash \sim A$ then since $\Gamma \mathscr{R} \Delta$, $\Delta \nvDash A$. But $\Delta \vDash A$, hence $\Gamma \nvDash A \vee \sim A$.

(2). $\vdash_I \sim\sim(A \vee \sim A)$.

A tableau proof for this is the following, where the reasons for the steps are obvious:

$$\begin{array}{l}\{\{F \sim\sim(A \vee \sim A)\}\},\\ \{\{T \sim(A \vee \sim A)\}\},\\ \{\{T \sim(A \vee \sim A), F(A \vee \sim A)\}\},\\ \{\{T \sim(A \vee \sim A), FA, F \sim A\}\},\\ \{\{T \sim(A \vee \sim A), TA\}\},\\ \{\{F(A \vee \sim A), TA\}\},\\ \{\{FA, F \sim A, TA\}\}.\end{array}$$

(3). $\nvdash_I \sim\sim A \supset A$.

The model of example (1) has the property that $\Gamma \vDash \sim\sim A$ but $\Gamma \nvDash A$.

(4). $\vdash_I (A \vee B) \supset \sim(\sim A \wedge \sim B)$.

The following is a proof:

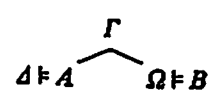
$$\begin{array}{l}\{\{F((A \vee B) \supset \sim(\sim A \wedge \sim B))\}\},\\ \{\{T(A \vee B), F \sim(\sim A \wedge \sim B)\}\},\\ \{\{T(A \vee B), T(\sim A \wedge \sim B)\}\},\\ \{\{T(A \vee B), T \sim A, T \sim B\}\},\\ \{\{T(A \vee B), FA, T \sim B\}\},\\ \{\{T(A \vee B), FA, FB\}\},\\ \{\{TA, FA, FB\}, \{TB, FA, FB\}\}.\end{array}$$

(5). $\nvdash_I \sim\sim(A \vee B) \supset (\sim\sim A \vee \sim\sim B)$.

A counter example is the following:

$$\begin{array}{l}\mathscr{G} = \{\Gamma, \Delta, \Omega\},\\ \Gamma \mathscr{R} \Gamma,\ \Delta \mathscr{R} \Delta,\ \Omega \mathscr{R} \Omega,\\ \Gamma \mathscr{R} \Delta,\ \Gamma \mathscr{R} \Omega\end{array}$$

$\Delta \vDash A$, $\Omega \vDash B$ is the $\vDash$ relation for atomic formulas, and $\vDash$ is extended as usual. We may schematically represent this model by



Key, facts in Δ; force facts back

Now $\Delta \models A$, so $\Delta \models A \vee B$. Likewise $\Omega \models A \vee B$. It follows that $\Gamma \models \sim\sim(A \vee B)$ But if $\Gamma \models \sim\sim A \vee \sim\sim B$, either $\Gamma \models \sim\sim A$ or $\Gamma \models \sim\sim B$. If $\Gamma \models \sim\sim A$, it would follow that $\Omega \models A$. If $\Gamma \models \sim\sim B$, it would follow that $\Delta \models B$. Thus $\Gamma \not\models \sim\sim A \vee \sim\sim B$.

(6) $\neg A \vee \neg\neg A$ does the model require branching