Applied Logic	**Lecture 2**
CS 486 Spring 2005	**Thursday, January 27, 2005**

Definition 1 *The relation "φ is an immediate subformula of ψ" is the smallest relation such that*

- *φ is an immediate subformula of $\neg\varphi$*
- *φ_1 and φ_2 are immediate subformulas of $\varphi_1 \wedge \varphi_2$*
- *φ_1 and φ_2 are immediate subformulas of $\varphi_1 \vee \varphi_2$*
- *φ_1 and φ_2 are immediate subformulas of $\varphi_1 \Rightarrow \varphi_2$.*

The relation "φ is a subformula of ψ" is the smallest relation such that

- *φ is a subformula of φ*
- *if φ is an immediate subformula of ψ, then φ is a subformula of ψ*
- *if φ is a subformula of ψ and ψ is a subformula of γ, then φ is a subformula of γ.*

The only formulas having no immediate subformulas are propositional variables (that is, φ is an immediate subformula of p never holds). Propositional variables are often called *atomic formulas*. Other formulas are often called *compound formulas*. We say a propositional variable p *occurs* in φ if p is a subformula of φ.

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Definition 2 *The* degree of a formula *is defined by the following (primitive) recursive function:*

$$
\begin{aligned}
degree(\varphi) = \mathbf{case}\ \varphi\ \mathbf{of}& \\
&\langle var, p\rangle \longrightarrow 0 \\
&\langle not, \psi\rangle \longrightarrow degree\psi + 1 \\
&\langle and, \psi_1, \psi_2\rangle \longrightarrow degree\psi_1 + degree\psi_2 + 1 \\
&\langle or, \psi_1, \psi_2\rangle \longrightarrow degree\psi_1 + degree\psi_2 + 1 \\
&\langle imp, \psi_1, \psi_2\rangle \longrightarrow degree\psi_1 + degree\psi_2 + 1 \\
\mathbf{end}.&
\end{aligned}
$$

For example, $p \wedge (q \vee \neg r)$ has degree 3, while $p \wedge (q \vee r)$ has degree 2.

Proposition 3 *φ is an atomic formula (i.e., a propositional variable) if and only if $degree(\varphi) = 0$.*

Proof. Immediate from the definition of degree. □

The degree of a formula lets us prove facts about the set $Form$ of all formulas by induction on the degree of formulas.

Proposition 4 *The Induction Principle for Formulas Let P be a property of formulas. If*

(i) $P(\varphi)$ holds for every formula of degree 0;

(ii) for all $n > 0$, if $P(\varphi)$ holds for every formula φ of degree $< n$, then $P(\varphi)$ holds for every formula of degree n;

Then $P(\varphi)$ holds for every formula φ.

Proof. Let X be the set $\{\varphi \mid P(\varphi) \text{ does not hold}\}$. We want to show that P holds for all formulas φ, i.e., that X is empty.

We proceed by contradiction. Assume X not empty. By a well-known property of the natural numbers, there exists a formula $\varphi_0 \in X$ that has minimal degree n_0, i.e., such that there is no formula in X with a smaller degree (there could be other formulas with the same degree). Let φ be an arbirary formula φ with degree $< n_0$. Since $degree(\varphi) < degree(\varphi_0)$, φ cannot be in X, therefore $P(\varphi)$ holds. Since φ was arbitrary, $P(\varphi)$ holds for all φs with degree less than n_0. By property (ii), then, this means that $P(\varphi_0)$ holds, i.e., $\varphi_0 \notin X$, a contradiction. Therefore, X is empty, as required. □

Proposition 5 *For every formula φ, the set $Sub(\varphi) = \{\psi \mid \psi \text{ is a subformula of } \varphi\}$ is finite.*

Proof. By using the Principle of Induction for Formulas.

First, we check the base case. If φ has degree 0, then φ is a propositional variable, and $Sub(\varphi) = \{\varphi\}$, which is finite.

Second, let $n > 0$, and assume for all formulas φ of degree $< n$, that $Sub(\varphi)$ is finite. Let φ be a formula of degree n. Since $n > 0$, φ is a compound formula, and thus either of the form $\neg\psi$, $\psi_1 \wedge \psi_2$, $\psi_1 \vee \psi_2$, or $\psi_1 \Rightarrow \psi_2$. If φ is $\neg\psi$, then $degree(\psi) = n - 1 < n$, therefore by induction hypothesis, $Sub(\psi)$ is finite. Since $Sub(\varphi) = Sub(\psi) \cup \{\varphi\}$, $Sub(\varphi)$ is finite. If φ is $\psi_1 \wedge \psi_2$, then $degree(\psi_1)$ and $degree(\psi_2)$ are both $< n$, and by the induction hypothesis, we have $Sub(\psi_1)$ and

$Sub(\psi_2$ finite; since $Sub(\varphi) = Sub(\psi_1) \cup Sub(\psi_2) \cup \{\varphi\}$, $Sub(\varphi)$ is finite. A similar argument works for $\vee$ and $\Rightarrow$. □

Assume a set $\mathbb{B} = \{t, f\}$ of *truth values*. Let S be a set of formulas.

Definition 6 *A valuation v on S is a function $v : S \to \mathbb{B}$.*

We say φ is true under valuation v if $v(\varphi) = t$. Similarly, φ is false under valuation v is $v(\varphi) = f$.

Definition 7 *A Boolean valuation v is a valuation on $Form$ such that:*

- $v(\neg\varphi) = t$ *if and only if* $v(\varphi) = f$
- $v(\varphi \wedge \psi) = t$ *if and only if* $v(\varphi) = t$ *and* $v(\psi) = t$
- $v(\varphi \vee \psi) = t$ *if and only if* $v(\varphi) = t$ *or* $v(\psi) = t$
- $v(\varphi \Rightarrow \psi) = t$ *if and only if when* $v(\varphi) = t$*, then* $v(\psi) = t$.

Given two valuations v_1, v_2, if $v_1(\varphi) = v_2(\varphi)$, then v_1 and v_2 *agree on* φ. If v_1 and v_2 agree on all formulas in a set S, then v_1 and v_2 *agree on* S.

Let S_1 and S_2 be sets of formulas with $S_1 \subseteq S_2$. If v_1 is a valuation on S_1, v_2 is a valuation on S_2, and v_1 and v_2 agree on S_1, then v_2 is an *extension* of v_1.

An *interpretation* v_0 is a valuation on propositional variables.

Proposition 8 *Let v_0 be an interpretation. If v and v' are Boolean valuations that extend v_0, then v and v' agree on all formulas.*

Proof. By induction on formulas. □

Thus, an interpretation can extend to *at most* a single Boolean valuation.

We can construct such a valuation explicitly:

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\begin{aligned}
value(\varphi, v_0) = \mathbf{case}\ \varphi\ \mathbf{of}\quad & \\
& \langle var, p\rangle \longrightarrow v_0(p) \\
& \langle not, \psi\rangle \longrightarrow vnot(value(psi, v_0)) \\
& \langle and, \psi_1, \psi_2\rangle \longrightarrow vand(value(\psi_1, v_0), value(\psi_2, v_0)) \\
& \langle or, \psi_1, \psi_2\rangle \longrightarrow vor(value(\psi_1, v_0), value(\psi_2, v_0)) \\
& \langle imp, \psi_1, \psi_2\rangle \longrightarrow vimp(value(\psi_1, v_0), value(\psi_2, v_0)) \\
& \mathbf{end}.
\end{aligned}
$$

where $vnot(t) = f$, $vnot(f) = t$; $vand(t,t) = t$, $vand(t,f) = vand(f,t) = vand(f,f) = f$; $vor(t,t) = vor(t,f) = vor(f,t) = t$, $vor(f,f) = f$; and $vimp(t,t) = vimp(f,t) = vimp(f,f) = t$, $vimp(t,f) = f$.

Proposition 9 *For every interpretation v_0, $value(-, v_0)$ is a Boolean valuation that extends v_0.*

Proof. By induction on formulas. □

Proposition 10 *For every interpretation v_0, there is a* unique *Boolean valuation that extends v_0, namely, $value(-, v_0$.*

Proof. Combining the previous two propositions. □

We often write $v_0 \models \varphi$ for $value(\varphi, v_0) = t$, and say that φ is true under interpretation v_0. Similarly, we write $v_0 \not\models \varphi$ for $value(\varphi, v_0) = f$, and say that φ is false under interpretation v_0.

Proposition 11 *Let φ be a formula. If the interpretations v_0 and v_0' agree on all propositional variables that occur in φ, then $value(\varphi, v_0) = value(\varphi, v_0')$.*

Proof. A straightforward induction will not quite work. We need to prove the slightly stronger statement: if the interpretations v_0 and v_0' agree on all propositional variables that occur in φ, then for all subformulas ψ of φ, $value(\psi, v_0) = value(\psi, v_0')$. Clearly, this implies the result we want, since φ is a subformula of φ. And establishing the stronger result is a simple application of induction on formulas. □