## 18.1 Decidability Issues

While the tableau method could be used as decision procedure for propositional logic, this will not work in first-order logic anymore. If a formula is not valid, the systematic method may lead to an infinite tableau. This is, however, not a deficiency of the tableau method. In fact, there is no correct and complete proof method for first-order logic that always terminates, as first-order logic is known to undecidable.

Nevertheless, in some cases the tableau method can decide that a formula is invalid although the proof is not finished yet. Whenever we have constructed a branch θ that represents a Hintkka set (over the finite domain of the parameters that occur on θ), then we know that the origin FX of the tableau is satisfiable and hence X must be invalid. In these rare cases, the Hintikka branch gives us a *counterexample* for the validity of the formula.

Example: Consider the invalid formula $[\forall x\, (Px \lor Qx)] \supset (\forall x\, Px\ \lor\ \forall x\, Qx)$.
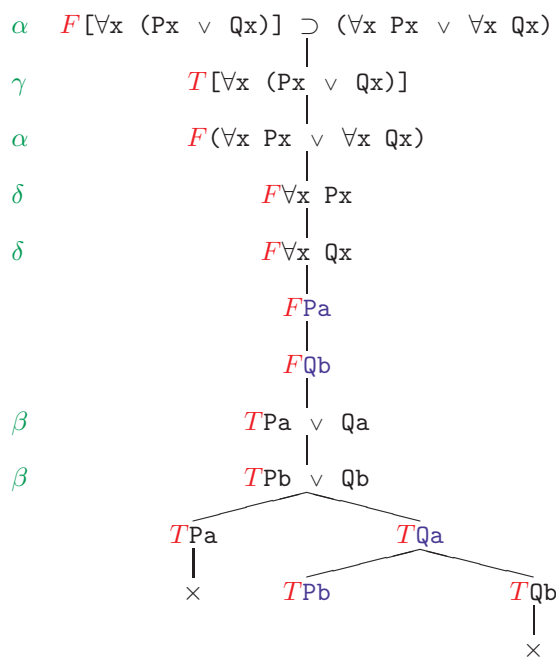
The tableau to the left cannot be extended anymore in any meaningful way and has one open branch θ, which is a Hintikka set for the 2 element domain $U = \{a,b\}$. In this branch the marked α, β, γ, and δ points are fulfilled for the domain U, since all formulas are true under the atomic valuation that assigns t to Qa and Pb and f to Qb and Pa.

In particular, the original formula $[\forall x\, (Px \lor Qx)] \supset (\forall x\, Px\ \lor\ \forall x\, Qx)$ evaluates to f under this interpretation.

It is possible to build this "Hintikka Test" into the tableau method and use it to prove that certain formulas cannot be valid. However, there are many formulas that are neither valid nor falsifiable in any finite domain. Any tableau proof attempt for these will run infinitely and at no stage of the proof will we know whether the formula is valid or not.

There are also formulas that are falsifiable, but not in any finite domain. However, the tableau method is "finite" and therefore not suited to produce counterexamples for them.

18.2 Compactness

The final important property of first-order logic that we have to investigate is compactness: *Given a set F of first-order formulas, what does the satisfiability of finite subsets tell us about the satisfiability of the whole set*. In propositional logic we have shown that a set S is uniformly satisfiable if all of its finite subsets are. We gave three proofs for that: one using tableau proofs and König's lemma, one giving a direct construction of a Hintikka set, and one using Lindenbaum's construction, extending S to a maximally consistent set, which turned out to be a proof set.

In first-order logic the question of compactness leads to a spin-off question. Can we extend Löwenheim's theorem to sets of formulas and prove "*if a set of formulas is satisfiable then it is satisfiable in a denumerable domain*"? This is the so-called *Skolem-Löwenheim theorem* and it follows from an extended form of compactness, which gives us uniform satisfiability over a denumerable domain if we know that all finite subsets are satisfiable? In the following we will use the tableau method to prove these facts.

Recall that in the propositional case we proved compactness by systematically constructing an tableau for the set S, using the fact that every finite subset of S is satisfiable to ensure that the tableau is infinite. We then used König's lemma to show that this tableau has an infinite branch, which in turn must be a Hintikka set. The construction of the tableau made sure that S is a subset of that set and hence satisfiable.

We will proceed in a similar fashoion for first-order logic. First, we define a *first-order tableau for a set S of pure formulas* (i.e formulas without parameters). Such a tableau starts with an arbitrary element of S at its origin and is then constructed by applying either one of the four rules α, , γ, or δ, or by adding another element of S to the end of an open branch. The elements of S so added are called the *premises* of the tableau. We call a tableau *complete* if every open branch is a Hintikka set for the universe of parameters and contains all the elements of S. Obviously every closed tableau is complete as well.

We first show that a complete tableau can be constructed for every set of first-order formulas.

Lemma 18.1 *For every denumerable set S there is a complete tableau for S.*

Proof. We construct the desired tableau by combining our systematic proof procedure with the construction of a tableau for S that we used in the propositional case. Arrange S as a denumerable sequence $X_1, X_2, X_3, \ldots, x_n, \ldots$.

We begin by placing X_1 at the origin of the tableau. This concludes stage 1. In stage $n+1$ we extend the tableau constructed at stage n as follows.

- If the tableau is already closed then stop. The formula is valid.
- Otherwise select a node Y in the tableau that is of *minimal level* wrt. the still unused nodes, extend *every* open branch θ through Y as in the systematic procedure, and add X_{n+1} to the end of every open branch.

By construction every open branch in the resulting tableau is a Hintikka set for the universe of parameters (we used the systematic method) and contains the set S. □

Using the method for constructing complete tableaux we can find an unsatisfiable finite subset for every set of formula that has a closed tableau.

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Lemma 18.2 *If a pure set S has a closed tableau, then a finite subset of S is unsatisfiable.*

Proof. Assume S has a closed tableau $\mathcal{T}$ and consider the set S_p of premises of $\mathcal{T}$. By König's lemma, $\mathcal{T}$ must be finite and so is S_p. S_p must be unsatisfiable, since otherwise every branch containing S_p would be open (recall that by construction the elements of the branch are derived from the formulas in S_p using tableau rules only.) $\qquad\square$

Now we use contraposition to prove "denumerable compactness.

Theorem 18.3

If all finite subsets of a denumerable set S of pure formulas are satisfiable, then S is uniformly satisfiable in a denumerable domain.

Proof. Let $\mathcal{T}$ be a complete tableau for S. Since all finite subsets of S are satisfiable, $\mathcal{T}$ cannot be closed due to the above lemma, so it has an open branch θ. Since $\mathcal{T}$ is complete, θ is a Hintikka set for the denumerable universe of parameters contained in S. Thus S is uniformly satisfiable in a denumerable universe. $\qquad\square$

Theorem 18.3 gives us the two desired results and another insight as immediate consequences.

Corollary 18.4 (Compactness of First-Order Logic)

If all finite subsets of a pure set S are satisfiable, then S is uniformly satisfiable

Corollary 18.5 (Skolem-Löwenheim theorem for First-Order Logic)

If a pure set S of is satisfiable then it is satisfiable in a denumerable domain.

Corollary 18.6

If no tableau for a pure set S can close, then S is satisfiable in a denumerable domain.

The last corollary leads to a lot of interesting results about theoretical properties of first-order logic that we won't discuss in this course. Those of you who are interested may study Smullyan's chapters VI and VII.