These are preliminary notes, containing only the necessary formalities. If I ever get around to it I will add more explanations

14.1 Assignments

Let Var be the type of propositional variables, and let $\mathbb{B} = \{f, t\}$ be the booleans (with f meaning false and t meaning true). An *assignment* is a function $v : Var \to \mathbb{B}$.

Given an assignment v, a boolean b, and a propositional variable p, the "updated" assignment $v|_b^p$ is the function (in $Var \to \mathbb{B}$) defined by

$$v|_b^p\ (q) = \begin{cases} b & \text{if } q = p \\ v(q) & \text{otherwise} \end{cases}$$

14.2 Semantics of $\mathbf{P^2}$

Let A be a $\mathbf{P^2}$-formula and let v be an assignment; let $v[A]$ (an abbreviation of $value(A, v)$) be the notation for the (boolean) value of A under v, and let $v[A] : \mathbb{B}$ be defined recursively as follows:

$$\begin{array}{lcl} v[\bot] & = & f \\ v[p] & = & v(p) \\ v[A \supset B] & = & (\neg_{\mathbb{B}}\, v[A])\ {}_{\mathbb{B}}\ v[B] \\ v[(\forall p)A] & = & (v|_f^p)[A]\ \wedge_{\mathbb{B}}\ (v|_t^p)[A] \end{array}$$

where $\neg_{\mathbb{B}} : \mathbb{B} \to \mathbb{B}$, ${}_{\mathbb{B}} : \mathbb{B} \times \mathbb{B} \to \mathbb{B}$, and $\wedge_{\mathbb{B}} : \mathbb{B} \times \mathbb{B} \to \mathbb{B}$ are the standard boolean operators.

For a finite set of formulas Γ, we define $v_\wedge[\Delta] = \bigwedge_{\mathbb{B}}\{\, v[A] \mid A \in \Delta \,\}$ and define $v_\vee[\Gamma] = \bigvee_{\mathbb{B}}\{\, v[A] \mid A \in \Gamma \,\}$, where $\bigwedge_{\mathbb{B}} S$ is the conjunction of the boolean values in the set S and $\bigvee_{\mathbb{B}} S$ is their disjunction. (By convention, $\bigwedge_{\mathbb{B}} \varnothing = t$ and $\bigvee_{\mathbb{B}} \varnothing = f$.) The value $v[\Delta \vdash \Gamma]$ of a sequent can now be defined as $(\neg_{\mathbb{B}}\, v_\wedge[\Delta])\ {}_{\mathbb{B}}\ v_\vee[\Gamma]$.

Examples: let $v(p_0) = t, v(p_1) = f, v(p_2) = f$

$$\begin{array}{rl} v[(p_0 \supset p_1)] = & (\neg_{\mathbb{B}}\, v[p_0])\ {}_{\mathbb{B}}\ v[p_1] = (\neg_{\mathbb{B}}\, t)\ {}_{\mathbb{B}}\ f = f \\ v[(p_0 \supset (p_0 \supset p_1))] = & (\neg_{\mathbb{B}}\, v[p_0])\ {}_{\mathbb{B}}\ v[p_0 \supset p_1] = (\neg_{\mathbb{B}}\, t)\ {}_{\mathbb{B}}\ f = f \\ v[(p_0 \supset p_0)] = & (\neg_{\mathbb{B}}\, v[p_0])\ {}_{\mathbb{B}}\ v[p_0] = (\neg_{\mathbb{B}}\, t)\ {}_{\mathbb{B}}\ t = t \\ v[(p_0 \supset (\forall p_0 (p_0 \supset p_0)))] = & (\neg_{\mathbb{B}}\, v[p_0])\ {}_{\mathbb{B}}\ (v|_f^{p_0})[p_0 \supset p_0]\ \wedge_{\mathbb{B}}\ (v|_t^{p_0})[p_0 \supset p_0] \\ = & f\ {}_{\mathbb{B}}\ (v[f \supset f])\ \wedge_{\mathbb{B}}\ (v[t \supset t]) = f\ {}_{\mathbb{B}}\ (t \wedge_{\mathbb{B}} t) = t \end{array}$$

The semantics of $\mathbf{P^2}$ can also be defined by *reducing* a $\mathbf{P^2}$-formula into an ordinary propositional formula. Since a variable can only assume two possible values, we can replace every universally quantified formula by $(\forall p)A$ by the formula $A[\top/p] \wedge [\bot/p]$, where $\top \equiv \bot \supset \bot$.[1]

[1]This reduction technique only works with $\mathbf{P^2}$. It cannot be used to reduce first-order logic to propositional logic,

14.3 Rules of P^2

The multiple-conclusioned sequent proof rules for $\mathbf{P^2}$ are as follows

$\bot L :$	$\Delta, \bot \vdash \Gamma$		
$\supset L :$	$\Delta, A \supset B \vdash \Gamma$ $\Delta \vdash A, \Gamma$ $\Delta, B \vdash \Gamma$	$\Delta \vdash A \supset B, \Gamma$ $\Delta, A \vdash B, \Gamma$	$\supset R$
$\forall L(B) :$	$\Delta, \forall p A \vdash \Gamma$ $\Delta, \forall p A, A[B/p] \vdash \Gamma$	$\Delta \vdash \forall p A, \Gamma$ $\Delta \vdash A[q/p], \Gamma$	$**$ $\forall R(q)$
$axiom :$	$\Delta, A \vdash A, \Gamma$		
$thinL :$	$\Delta, A \vdash \Gamma$ $\Delta \vdash \Gamma$	$\Delta \vdash A, \Gamma$ $\Delta \vdash \Gamma$	$thinR$

$**$ this is only legal if $q \notin FV(\Delta, \Gamma, \forall p A)$.

The rules for $\exists$ can be derived from the rules given above:

$\exists L :$	$\Delta, \exists p A \vdash \Gamma$ $\Delta, A|^p_q \vdash \Gamma$ $**$	$\Delta \vdash \exists p A, \Gamma$ $\Delta \vdash A|^p_B, \Gamma$	$\exists R$

The familiar rules for $\wedge$, $\vee$, and $\sim$ can also be derived.

An example proof:

$\vdash (\forall p.p) \supset \bot$	
$\forall p.p \vdash \bot$	$\supset R$
$\bot \vdash \bot$	$\forall L(\bot)$

Here is a proof that the two definitions of conjunction given above are actually equivalent.

$\vdash A \wedge B \supset (\forall p)((A \supset B \supset p) \supset p)$	$\supset R$
$A \wedge B \vdash (\forall p)((A \supset B \supset p) \supset p)$	$\forall R(P)$
$A \wedge B \vdash (A \supset B \supset P) \supset P$	$\supset R$
$A \wedge B, (A \supset B \supset P) \vdash P$	$\supset L$
1.$A \wedge B \vdash A, P$	$\wedge L$
$A, B \vdash A, P$	$axiom$
2.$A \wedge B, B \supset P \vdash P$	$\supset L$
2.1.$A \wedge B \vdash B, P$	$\wedge L$
$A, B \vdash B, P$	$axiom$
2.2.$A \wedge B, P \vdash P$	$axiom$

since variables may assume infinitely many values.

$\vdash (\forall p)((A \supset B \supset p) \supset p) \supset A \wedge B$	$\supset R$
$(\forall p)((A \supset B \supset p) \supset p) \vdash A \wedge B$	$\forall L(A)$
$(\forall p)((A \supset B \supset p) \supset p), (A \supset B \supset A) \supset A \vdash A \wedge B$	$\supset L$
$1. (\forall p)((A \supset B \supset p) \supset p) \vdash A \supset B \supset A, A \wedge B$	$\supset R$
$(\forall p)((A \supset B \supset p) \supset p), A \vdash B \supset A, A \wedge B$	$\supset R$
$(\forall p)((A \supset B \supset p) \supset p), A, B \vdash A, A \wedge B$	*axiom*
$2. (\forall p)((A \supset B \supset p) \supset p), A \vdash A \wedge B$	$\forall L(B)$
$((A \supset B \supset B) \supset B), A \vdash A \wedge B$	$\supset L$
$2.1. A \vdash A \supset B \supset B, A \wedge B$	$\supset R$
$A, A \vdash B \supset B, A \wedge B$	$\supset R$
$A, A, B \vdash B, A \wedge B$	*axiom*
$2.2. B, A \vdash A \wedge B$	$\wedge R$
$2.2.1. B, A \vdash A$	*axiom*
$2.2.2. B, A \vdash B$	*axiom*