(1) Prove that any closed tableau can be further extended into an atomically closed tableau.

Hint: Show by induction on the degree of a formula X that any set S that contains both X and $\bar{X}$ has an atomically closed tableau.

Proposition 1. *Any closed tableau can be further extended into an atomically closed tableau.*

Proof. Suppose a closed branch of the tableau contains a signed formula X and its conjugate $\bar{X}$. It suffices to show that this branch can be extended to contain atomic conjugates.

Without loss of generality, assume $X = T(\phi)$ for some formula ϕ. We proceed by induction on the degree of ϕ.

Base case. If ϕ has degree 0, then ϕ is a propositional variable, and the branch already contains atomic conjugates.

Inductive step. Let $n \geq 0$ and assume for all formulæ ϕ of degree at most n, the branch can be extended to contain atomic conjugates. Let ϕ be a formula of degree $n+1$. ϕ must be a compound formula, and thus one of the following forms.

- $\sim \psi$. Using $T(\sim \psi)$ and $F(\sim \psi)$ derives $F(\psi)$ and $T(\psi)$, respectively. Since ψ has degree n, by the induction hypothesis, this branch can be further expanded to contain atomic conjugates.
- $\psi_1 \wedge \psi_2$, $\psi_1 \vee \psi_2$, or $\psi_1 \supset \psi_2$. We can derive:

$T(\psi_1 \wedge \psi_2)$	$F(\psi_1 \vee \psi_2)$	$F(\psi_1 \supset \psi_2)$
$T(\psi_1)$	$F(\psi_1)$	$T(\psi_1)$
$T(\psi_2)$	$F(\psi_2)$	$F(\psi_2)$
$F(\psi_1 \wedge \psi_2)$	$T(\psi_1 \vee \psi_2)$	$T(\psi_1 \supset \psi_2)$
$F(\psi_1) \mid F(\psi_2)$	$T(\psi_1) \mid T(\psi_2)$	$F(\psi_1) \mid T(\psi_2)$

  In all cases, both resulting branches have either $T(\psi_1)$ and $F(\psi_1)$ or $T(\psi_2)$ and $F(\psi_2)$. Since both ψ_1 and ψ_2 have degree at most n, by the induction hypothesis, both branches in each case can be further expanded to contain atomic conjugates.

Hence, by the principal of induction on formulæ, any closed branch of a tableau can be further extended to contain atomic conjugates. □

(2) Let S be a set of formulas such that for any interpretation v_0 there is a formula $X \in S$ with $value(X, v_0) = t$. Show, using the compactness theorem, that there is a finite subset $\{X_1, .., X_n\}$ of S such that $X_1 \vee .. \vee X_n$ is a tautology.

Proposition 2. *Suppose S is a set of formulæ such that for any interpretation v_0, there is a formula $X \in S$ with $\mathsf{value}(X, v_0) = \mathsf{t}$. Then there is a finite subset $\{X_1, \ldots, X_n\} \subset S$ such that $X_1 \vee \ldots \vee X_n$ is a tautology.*

Proof. Proof by contradiction. Suppose, for all finite subsets $\{X_1, \ldots, X_n\} \subset S$, $X_1 \vee \ldots \vee X_n$ is not a tautology. i.e., $\sim (X_1 \vee \ldots \vee X_n)$ is satisfiable. Then $\sim X_1 \wedge \ldots \wedge \sim X_n$ is satisfiable.

Let $\tilde{S} = \{\sim X : X \in S\}$. By the compactness theorem, $\tilde{S}$ is satisfiable. Thus, let v_0 be an interpretation such that $\forall \sim X \in \tilde{S}.\ \mathsf{value}(\sim X, v_0) = \mathsf{t}$. Then we have $\forall X \in S.\ \mathsf{value}(X, v_0) = \mathsf{f}$. Thus, we have found a valuation for which no formula $X \in S$ has $\mathsf{value}(X, v_0) = \mathsf{t}$, contradicting our original assumption about S. □

(3) A set of formulas S is *complete* if every formula or its negation belongs to S. A (deductively closed) *theory* is a consistent set of formulas T such that every formula deducible from T belongs to T.

Prove that every theory is the intersection of all its complete consistent extensions.

Proposition 3. *Every theory is the intersection of all its complete consistent extensions.*

Proof. Let T be a theory, and let $\mathcal{E}$ be the set of complete consistent extensions of T.

Clearly, $T \subseteq \bigcap \mathcal{E}$. Suppose $X \notin T$. We now show $X \notin \bigcap \mathcal{E}$.

According to Smullyan p.40, if $\sim X$ is inconsistent with T, then X is deducible from T. Since $X \notin T$, X must not be deducible from T; thus, $T \cup \{\sim X\}$ is consistent. This means that there must be some $S \in \mathcal{E}$ with $T \cup \{\sim X\} \subseteq S$. Since S is complete and consistent, we must then have $X \notin S$. Thus, $X \notin \bigcap \mathcal{E}$. □