20 Mar 2024

# NP-Complete Graph Problems

Announcement: PSet 7 to be released Friday morning.

Q1 will have optional "early hand-in."

due Tues. 4pm

promise to grade before end of Sp Brk.

solution set for entire PSet 7

will be released to all

(couple of days ~~after~~ ~~latest~~ slip-day deadline)

regardless of early Q1 option.

Rest of PSet 7 has usual Thurs night deadline.

**Recall.**

3SAT. Given Boolean variables $x_1, \ldots, x_n$

forming literals $x_1, \overline{x_1}, x_2, \overline{x_2}, \ldots, x_n, \overline{x_n}$

and clauses $C_1, \ldots, C_m$

each is disjunction (Boolean OR) of $\leq 3$ literals.

... does a truth assignment satisfying all clauses exist?

IND SET. Given undirected graph $G$

pos. integer $k$

... does exist set $S$ of $k$ vertices, s.t. every edge has $\leq 1$ endpoint in $S$?

Claim. $3SAT \leq_p$ IND SET

"Given an algorithm that solves IND SET

we can make an efficient 3SAT algorithm."

Goal. ① Transform input of 3SAT to input of IND SET.

② Transformation runs in poly time. (In fact, will be linear time.)

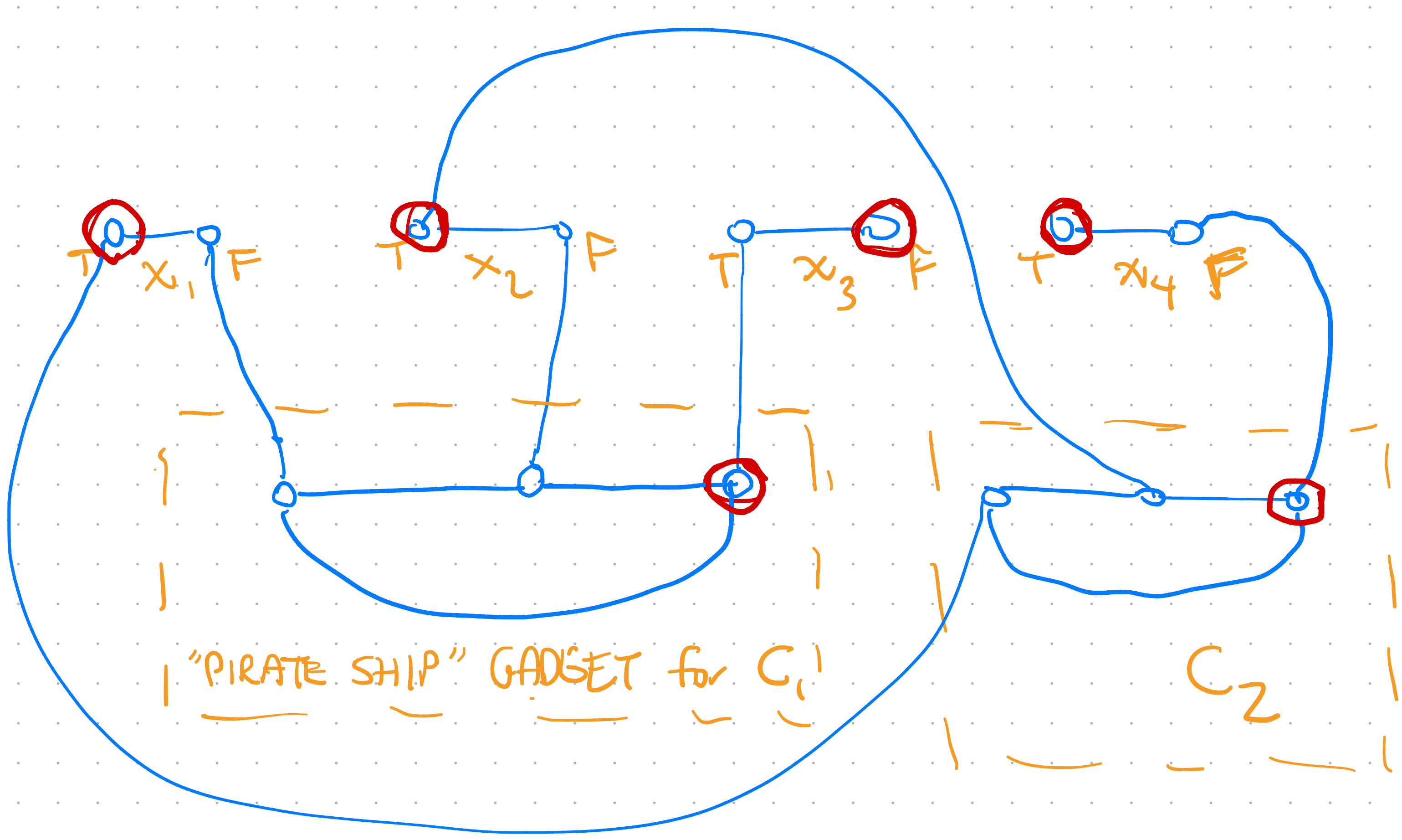
③ If 3SAT input has a satisfying truth assignment, the IND SET instance will have a k-element ind set.

START HERE.

Represent constituent units of 3SAT problem using "gadgets!" (subgraphs)

④ If 3SAT input has no satisfying assign. the IND SET instance will have no k-element independent set.

$C_1: x_1 \vee x_2 \vee \overline{x_3}$ $C_2: \overline{x_1} \vee \overline{x_2} \vee x_4$



In general, the reduction takes

Variables $\longrightarrow$ vertices ($2n$ vert's)

$x_1, \ldots, x_n$ $\longrightarrow$ $u_1, \ldots, u_n$ (T)

$v_1, \ldots, v_n$ (F)

Clauses $\longrightarrow$ $w_{ij}$ ($\leq 3m$ verts) $\forall$ pairs $i,j$ such $x_j$ or $\overline{x_j}$ appears in clause $i$.

<u>edge set:</u> Connect $(u_j, v_j)$ $(n)$ $\forall j \in [n]$

$(w_{ij}, w_{ik})$ $(\leq 3m)$ $\forall j \neq k$ s.t. $x_j, x_k$ both present in $C_i$ (or $\overline{x_j}, \overline{x_k}$).

gadget internal edges

$(w_{ij}, u_j)$ if $\overline{x_j}$ is in $C_i$

$(w_{ij}, v_j)$ if $x_j$ is in $C_i$.

$\leq 3m$

Set target indep set size, $k$, to be $n+m$.

Running time: $O(2n+3m+n+6m) = O(m+n)$.

<u>CLIQUE:</u> Given (undir.) graph $G$ and $k \in \mathbb{N}$

does $G$ have a set of $k$ vertices, $S$, s.t. every two elements of $S$ are connected by an edge?

VERTEX COVER: Given $G$ & $k$

does $G$ have a set of vertices, $S$, with $k$ elements, that "covers" every edge? ($S$ contains at least one endpoint of every edge.)

IND SET $\leq_p$ CLIQUE

Given $G = (V, E)$ and $k$

construct $\overline{G} = (V, \overline{E})$ and $k$.

$S$ is indep set in $G \iff$
$S$ is clique in $\overline{G}$.