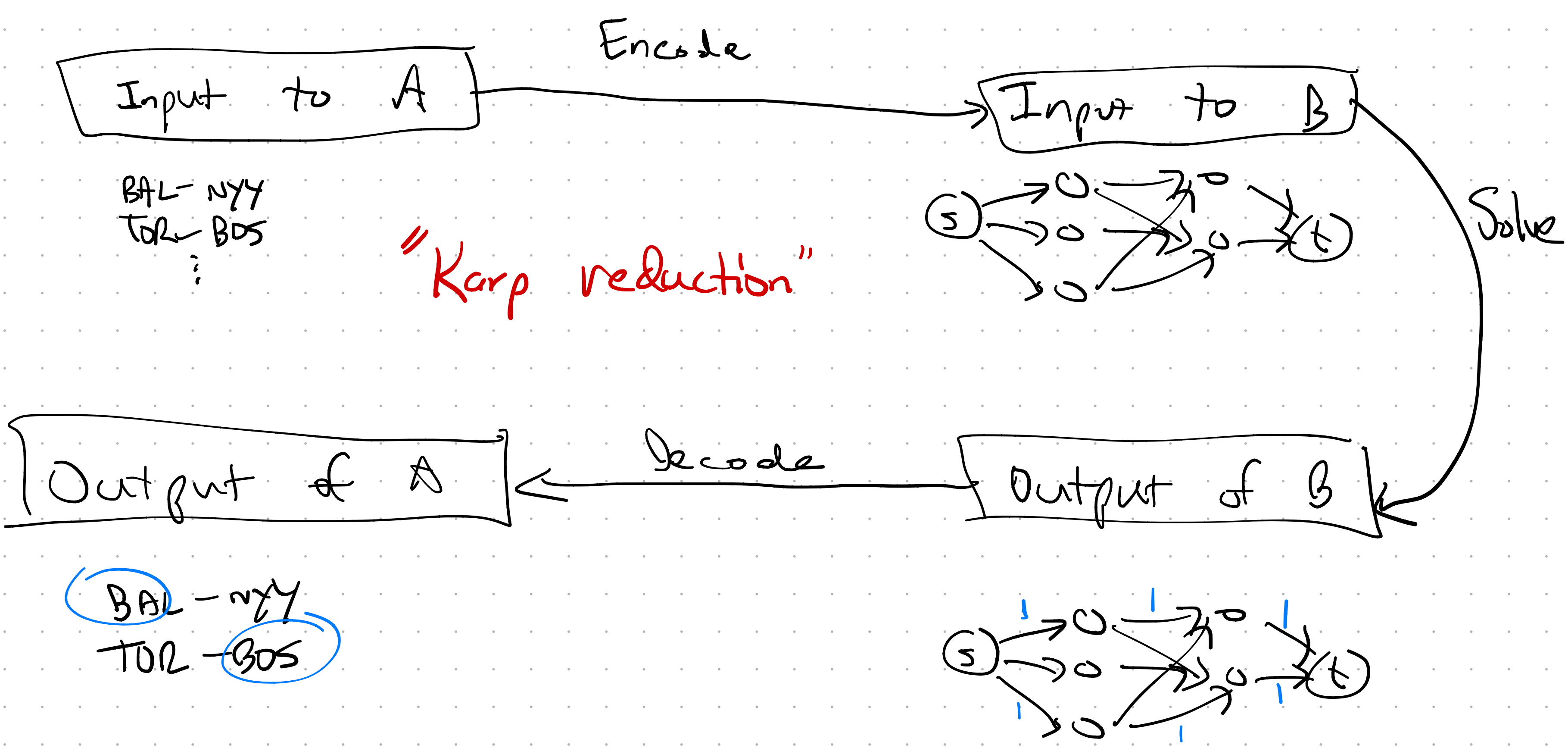
18 Mar 2024

NP Completeness (Intro)

Announcement: today (Mar 18) is the last date to drop classes.

Some problem you want to solve (A)

Some problem you already can solve (B)



When $\exists$ a Karp reduction from A to B running in polynomial time, we ~~denote~~ that relation as

$$A \leq_p B$$

and interpret this as "if we can solve B then we can solve A," or, "A is at least as easy to solve as B."

Equivalently: If we cannot solve A (efficiently) then we also cannot solve B (efficiently).

The "root of hardness" (one problem assumed by most people to be computationally hard) is SAT. ("Boolean satisfiability")

<u>Given:</u> variables $x_1, \ldots, x_n$ taking $\{T, F\}$ values.

clauses $C_1, \ldots, C_m$ each a disjunction of 2 or more literals.

disjunction = Boolean OR

("literal" = "variable or its negation")

e.g. $C_1 = x_2 \vee \overline{x_4} \vee x_5 \vee \overline{x_8} \vee x_9$

<u>Question:</u> Does there exist a truth assignment for $x_1, \ldots, x_n$ that satisfies every clause

<u>Ex.</u>

$$\underbrace{(x_1 \vee x_2)}_{C_1} \wedge \underbrace{(x_2 \vee x_3)}_{C_2} \wedge \underbrace{(\overline{x_3} \vee \overline{x_1})}_{C_3}$$

$x_1 = T$
$x_2 = T$
$x_3 = F$

$x_1 = \overline{x_2}$ $\quad$ $x_2 = \overline{x_3}$

$$\boxed{(x_1 \vee x_2) \wedge (\overline{x_1} \vee \overline{x_2})} \wedge \boxed{(x_2 \vee x_3) \wedge (\overline{x_2} \vee \overline{x_3})}$$
$$\wedge \boxed{(x_3 \vee x_1) \wedge (\overline{x_3} \vee \overline{x_1})}$$

$x_3 = \overline{x_1}$

Not satisfiable!

<u>Status of SAT:</u> We don't know any algorithm faster than $O(2^{n - o(n)})$.

Even solving in $O(1.99^n)$ time would be a major breakthrough.

<u>3-SAT:</u> The special case of SAT where each clause has 3 literals.

<u>Known:</u> $\text{SAT} \leq_p \text{3-SAT}$

<u>Believed:</u> 3-SAT requires $\geq c^n$ running time for some $c > 1$.

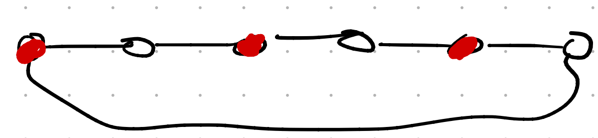
INDEPENDENT SET.

Given: Graph G (undirected)

$k \in \mathbb{N}$

Question: Does G have a set of k vertices with no edge joining any two of them?

(Such a vertex set is called an "independent set".)

Ex. $G =$  $k=3$ Yes

$G =$  $k=3$ No

Claim: INDEPENDENT SET is at least as hard as 3-SAT.

$$\text{3-SAT} \leq_p \text{IND SET}$$

$\boxed{\text{Input to 3 SAT}} \xrightarrow{\text{Encode}} \boxed{\text{Input to IND SET}}$

"How can we transform a 3-SAT problem to an independent set problem, so that solving the IND SET instance gives you the answer to 3-SAT?"