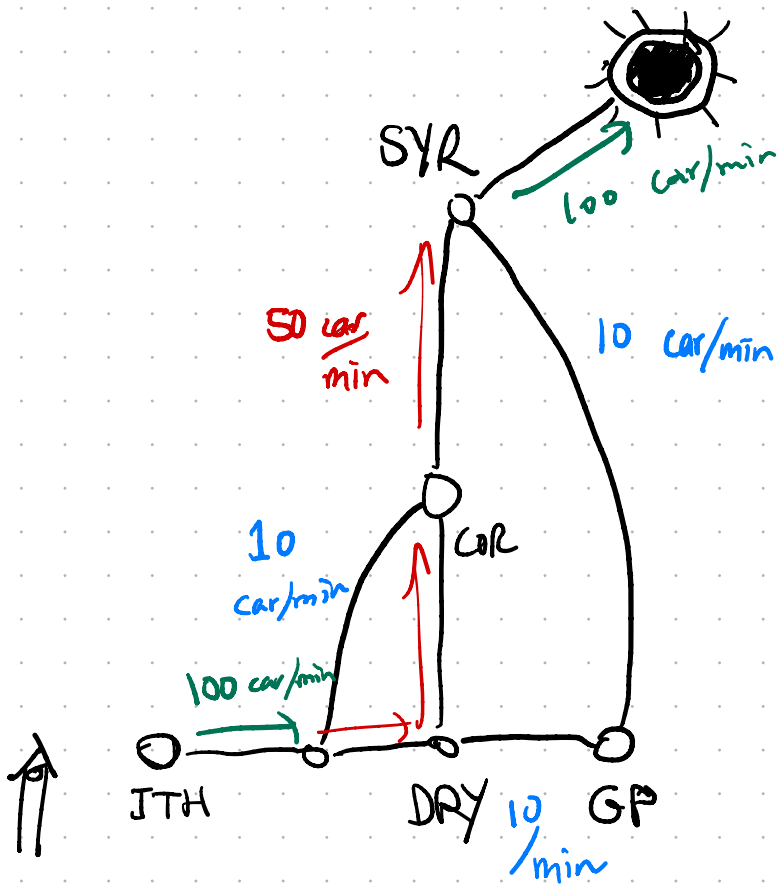
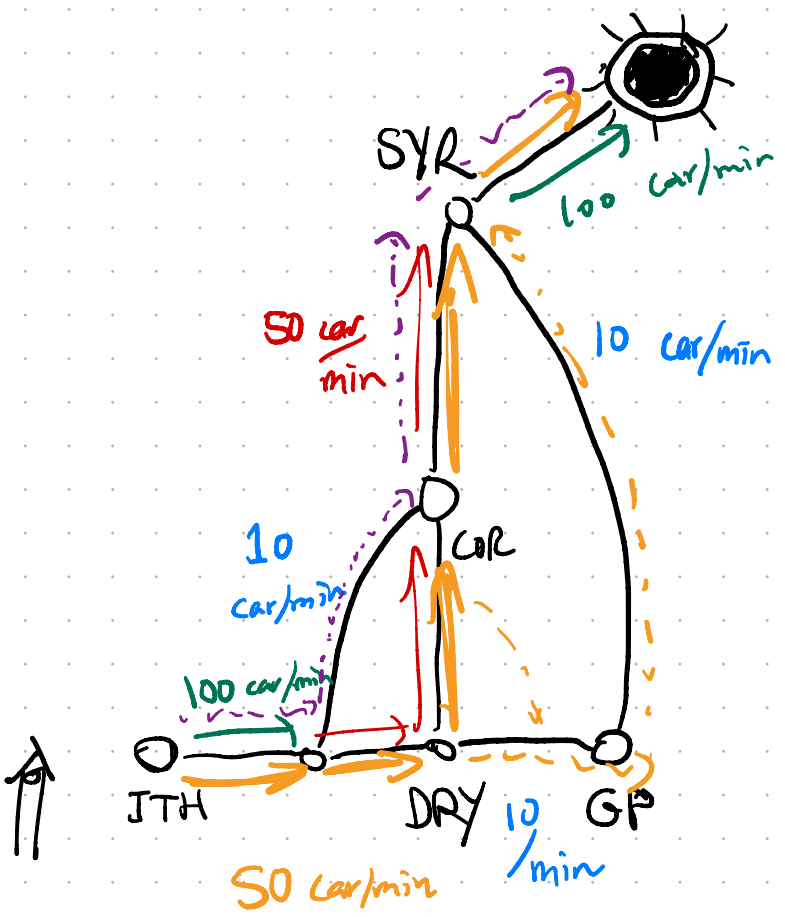
4 Mar 2024 Network Flow (§7.1)

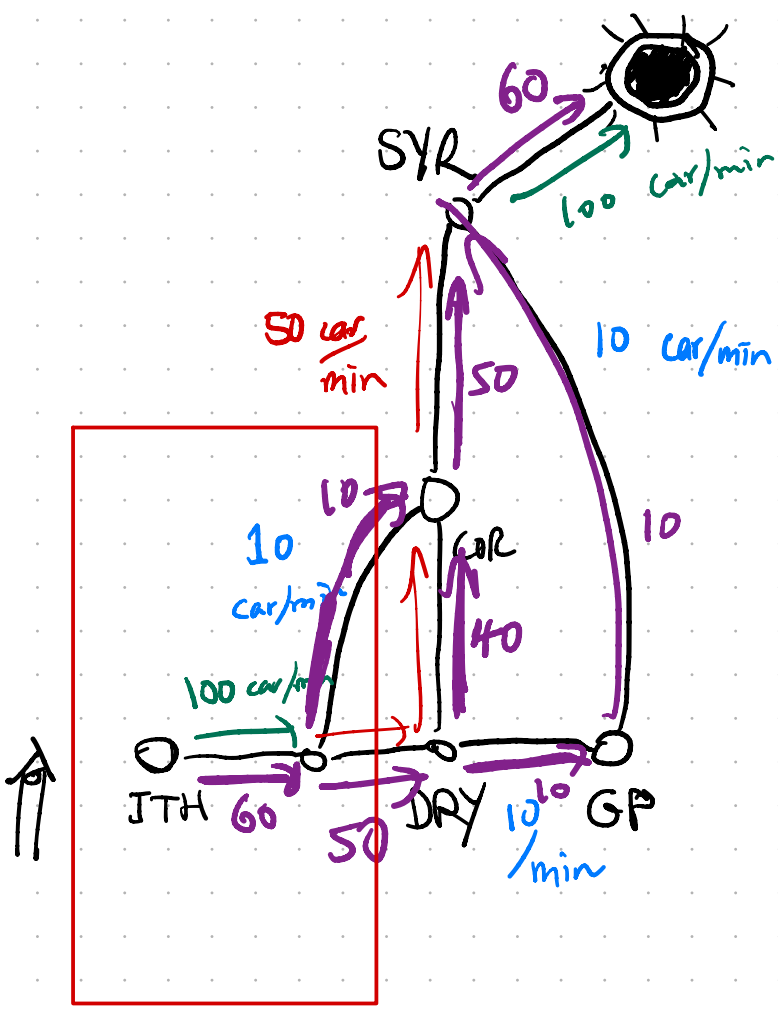
Announcements: To be posted very soon....

- prelim solutions
- Homework 3 grades



Max rate at which cars can get from ITH to eclipse? (in cars/min)





60 is the max rate of cars exiting the red box because the only two roads leaving it have combined capacity 60.

GLOSSARY: "network flow" — a class of algo problems involving sending stuff around a capacitated network

"flow network" — an instance of network flow

"flow" — a solution for a particular flow network

"max flow" — an optimal solution.

Def. A flow network (G, s, t, c) is:

- directed graph G
- vertices $s \neq t$ in $V(G)$ called "source" and "sink"
- capacities $c(e) \geq 0$ for all $e \in E(G)$.

  $c(e) \in \mathbb{R}_{\geq 0} \cup \{\infty\}$
- Conventions (for convenience, not necessity)
  - no edges into s or out of t
  - every vertex belongs to at least one edge. $\Rightarrow$ $|V| \leq 2|E|$, so for example $O(|V| + |E|) = O(|E|)$.

Def. A **flow** in a flow network is a function $f : E \to \mathbb{R}_{\geq 0}$ satisfying

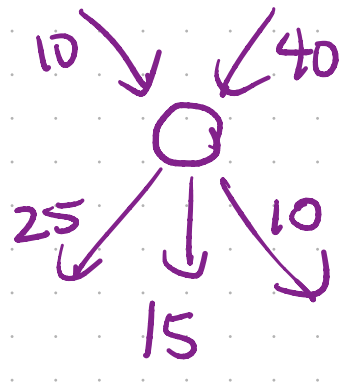
(1) [conservation] for all $v \neq s, t$

$$f^{in}(v) = f^{out}(v)$$

where

$$f^{in}(v) = \sum_{(u,v) \in E} f(u,v)$$

$$f^{out}(v) = \sum_{(v,w) \in E} f(v,w)$$



(2) [capacity] for all $e \in E(G)$

$$0 \leq f(e) \leq c(e)$$

Def. The value of a flow, f, is

$$v(f) = f^{out}(s) = f^{in}(t)$$

↑ must be equal because of flow conservation

A maximum flow in a flow network (G, s, t, c) is a flow f that attains the maximum value of $v(f)$.

Residual capacities and augmenting paths

Def. If G is a flow network and f is a flow, there is a residual graph called G_f with

- vertices $V(G_f) = V(G)$
- edges $\{(u,v) \mid (u,v) \in E(G) \text{ and } f(u,v) < c(u,v)\}$

  "forward edges"

  $\cup$ $\{(v,u) \mid (u,v) \in E(G) \text{ and } f(u,v) > 0\}$

  "backward edges"

These have "residual capacities"

$$c_f(u,v) = c(u,v) - f(u,v) \quad \text{for forward edge}$$

$$c_f(v,u) = f(u,v) \quad \text{for backward edge}$$

An <u>augmenting path</u> is an s-t path in G_f.