31 January. More Greedy Algos.

Plan

1. Finish Greedy Stays Ahead Analysis
2. Announcements
3. Minimum Spanning Tree Problem

Earliest Finish Time

1. Sort jobs by finish time
2. Schedule = $\{\}$

Iterate through jobs in sorted order $j = 1 \dots n$

- if job j does not conflict w/ Schedule
  - $\hookrightarrow$ Schedule $\leftarrow$ Schedule $\cup \{j\}$

Return Schedule.

Optimal schedule $O^* = \langle [s_1^*, f_1^*], [s_2^*, f_2^*], \ldots, [s_\ell^*, f_\ell^*] \rangle$

EFT schedule $S = \langle [s_1, f_1], [s_2, f_2], \ldots, [s_k, f_k] \rangle$

"If EFT stays ahead, then EFT is optimal"

Earliest Finish Time Lemma

For all $j \in \{1, \ldots, k\}$

$$f_j \leq f_j^* \implies S \text{ is also optimal.}$$

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By <u>induction</u> on jobs added by EFT.

<u>Base Case</u>. EFT adds jobs by earliest finish time.

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$\Longrightarrow$ $f_1 \leq f_1^*$ ✓

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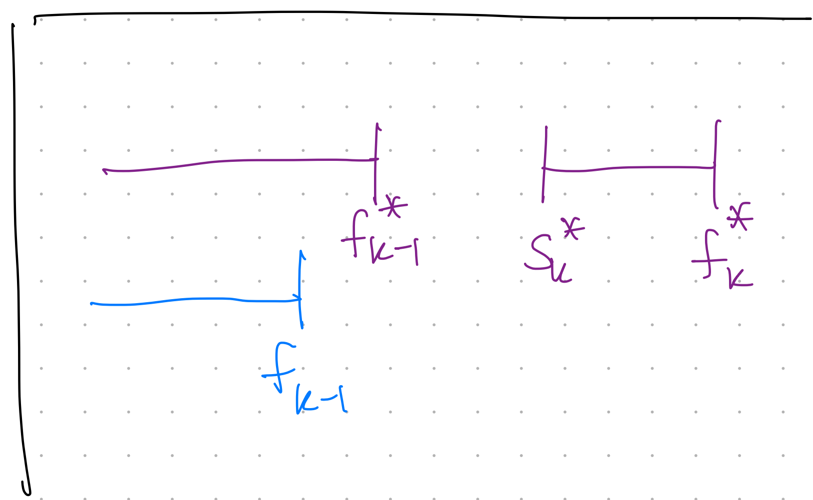
Inductive Hypothesis. For all $i < k$, $\quad f_i \le f_i^*$

By IH, $\underbrace{f_{k-1} \le f_{k-1}^*}_{\text{By IH}} < s_k^*$. $\quad \Rightarrow \quad f_{k-1} < s_k^*$



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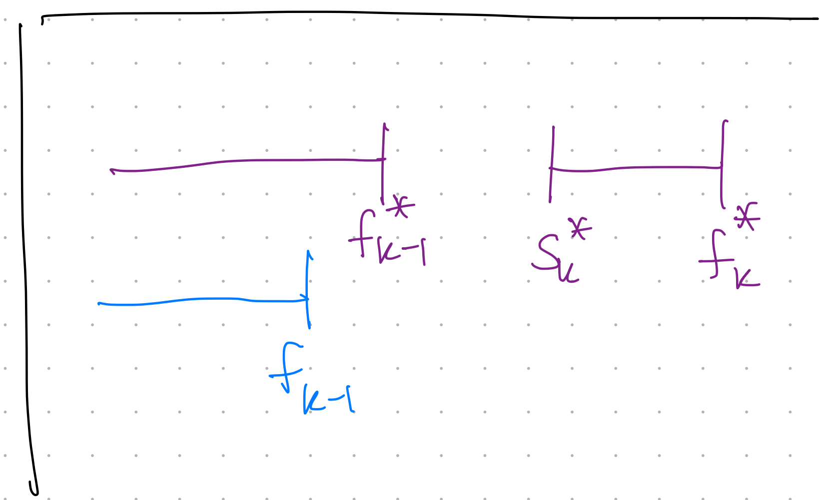

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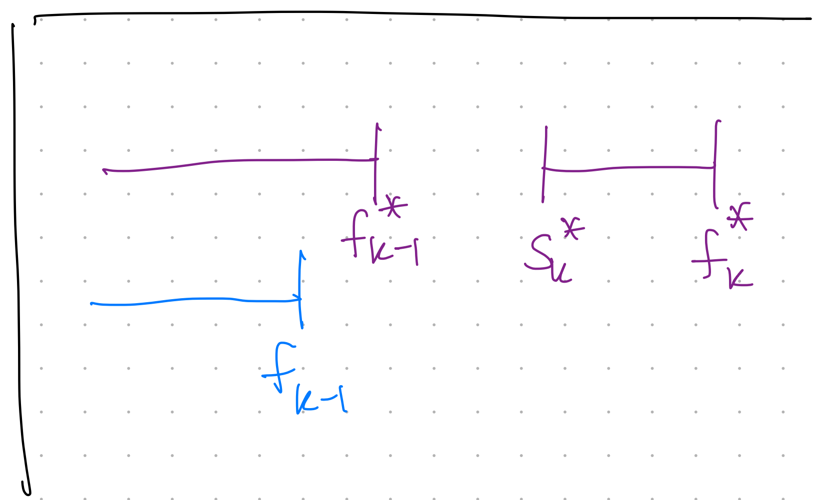

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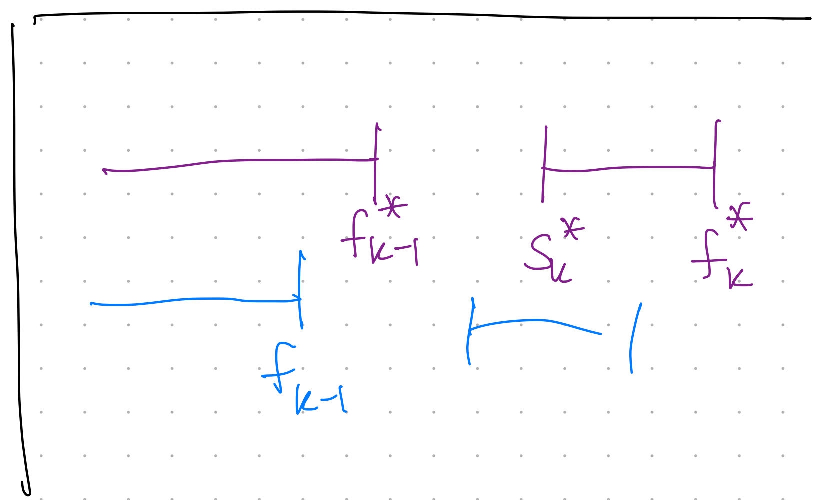

"If EFT stays ahead, then EFT is optimal"

Earliest Finish Time Lemma

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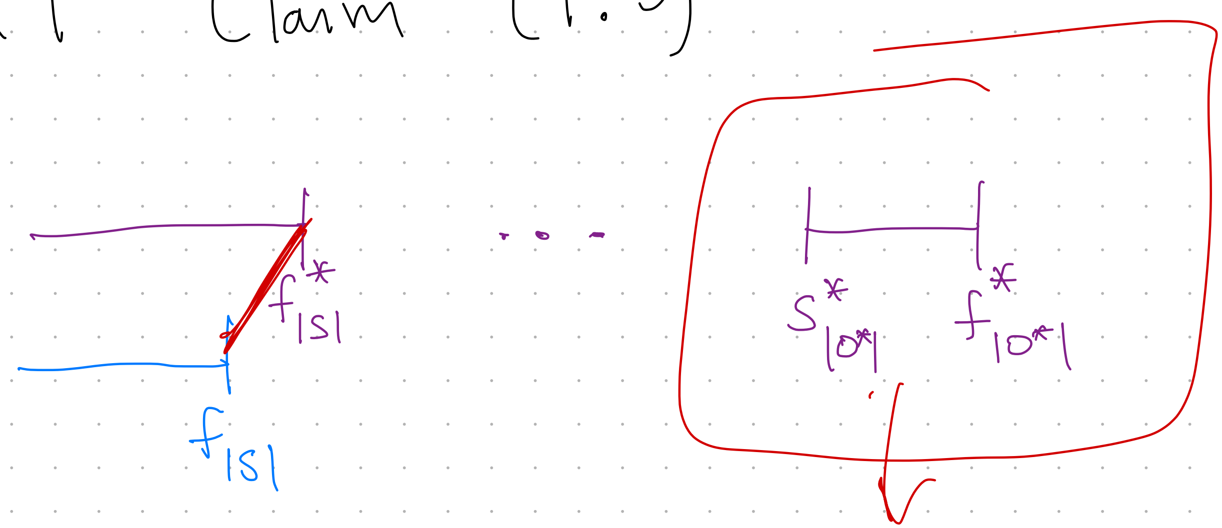
Similar Argument.

See KT Claim (4.3)



Announcements

* Enrollment cap lifted
* HW 1 due Thurs, 11:59 PM
  ↳ At most 3 slip days (Sun, 11:59 PM)

Greedy Graph Algorithms

Minimum Spanning Tree

Given a connected, undirected, weighted, graph $G = (V, E, w)$

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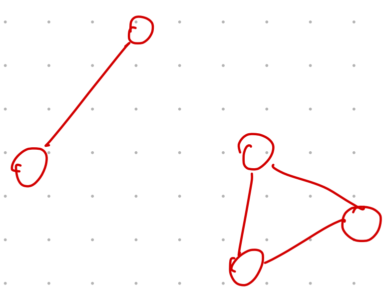
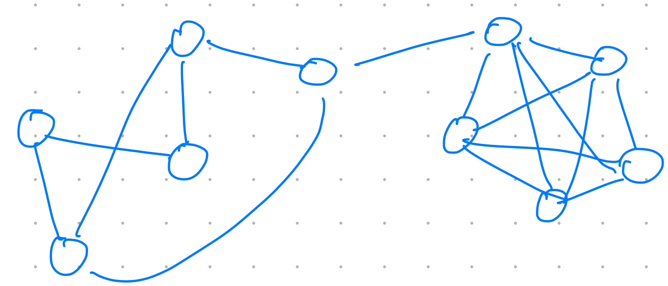


Not connected



Connected

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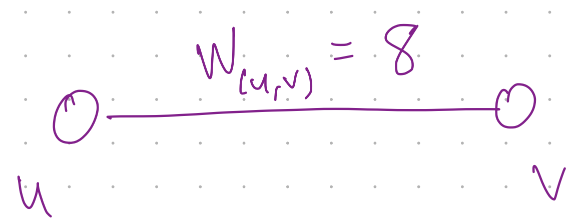
- A graph is undirected if $(u,v) \in E \Leftrightarrow (v,u) \in E$

That is, (u,v) and (v,u) represent the same edge

Minimum Spanning Tree

Given a connected, undirected, weighted graph $G = (V, E, w)$

- A graph is connected if $\forall u, v \in V$ $\exists$ a path $p = (e_1, \dots, e_{|p|})$ connecting u & v.
- A graph is undirected if $(u,v) \in E \iff (v,u) \in E$
- Each edge $e \in E$ has an associated Nonnegative weight $w_e \geq 0$



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Minimum Spanning Tree

Given a connected, undirected, weighted graph $G=(V,E,W)$

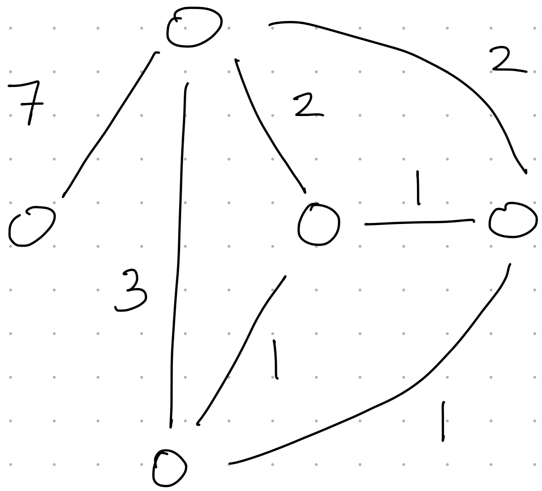
Find a minimum spanning tree $T=(V,E'\subseteq E)$

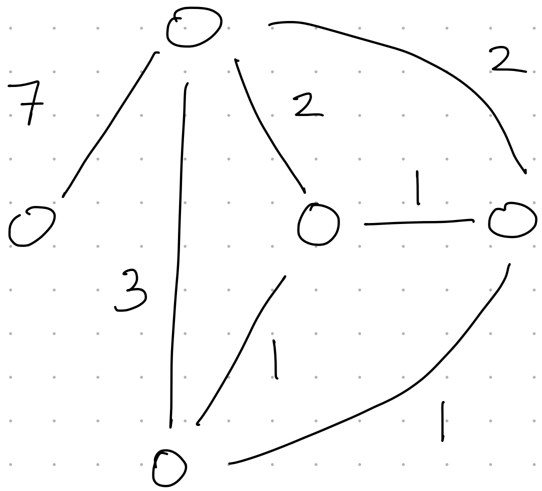
minimum → minimum total weight

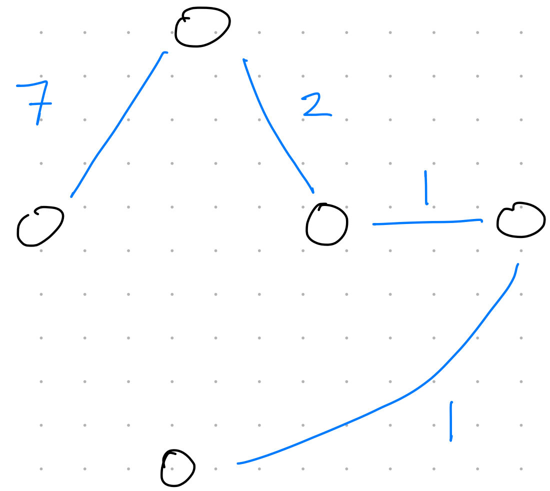
spanning → connects all $u,v \in V$

tree → graph with no cycles

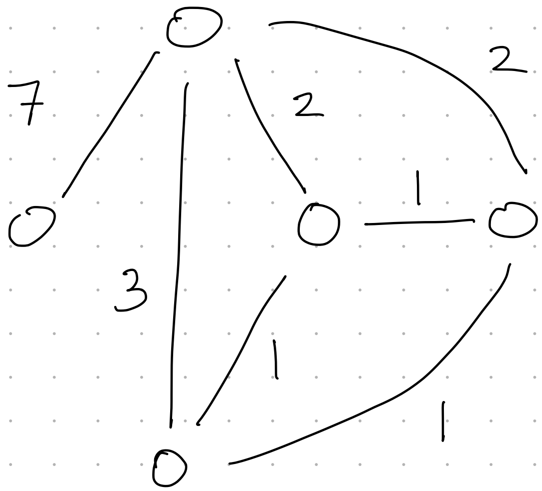
$$W_T = \sum_{e \in T} w_e$$

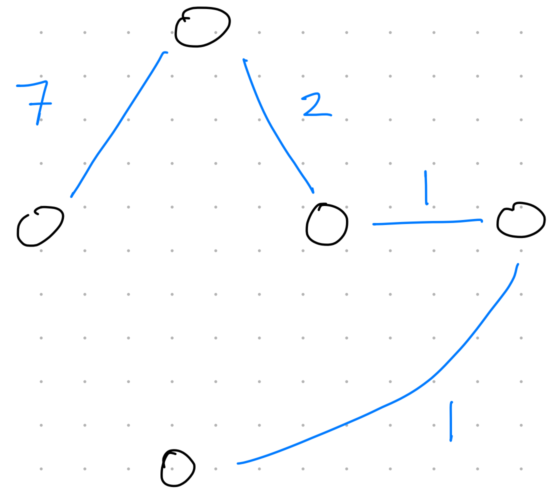

7
2
2
1
3
1
1





$W_T = 11$





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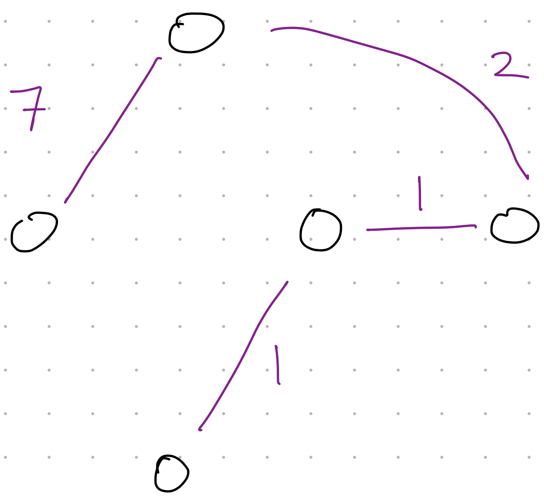

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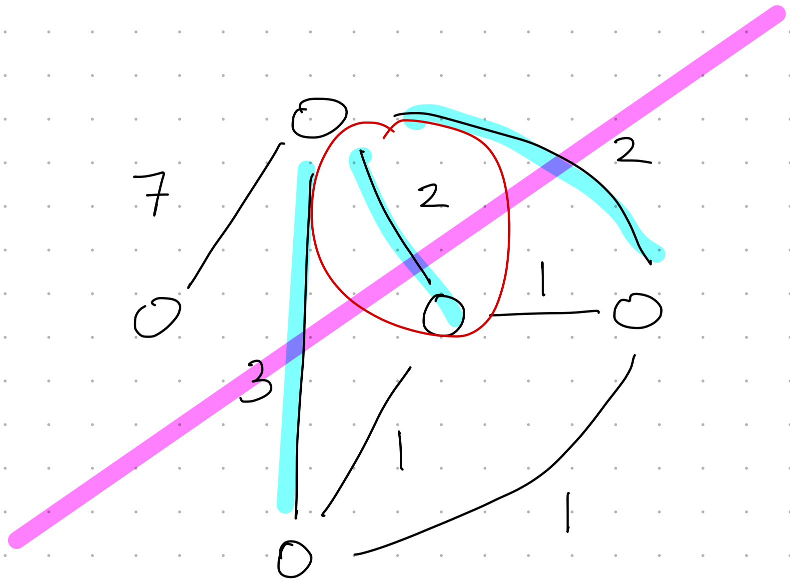
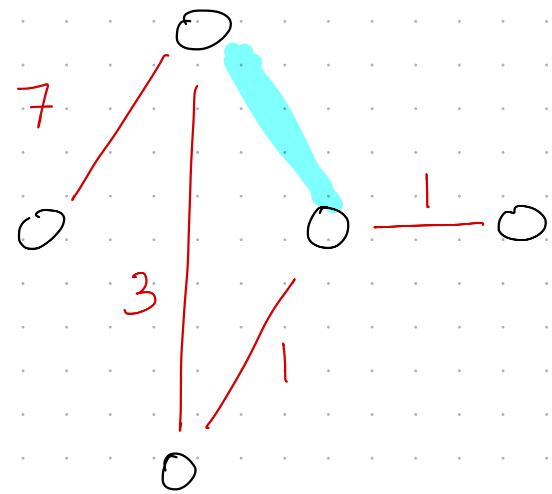


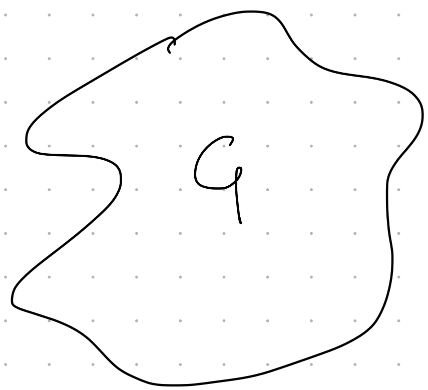


$W_T = 12$

Given a tree T, how do we know that

T is / is not a min spanning tree?

Cut Lemma. Given a graph, $G = (V, E, W)$.



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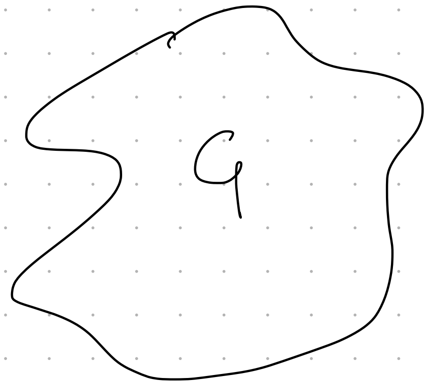

Simplifying Assumption

Edge weights are distinct.

$\forall e, e' \in E, \quad w_e \neq w_{e'}$

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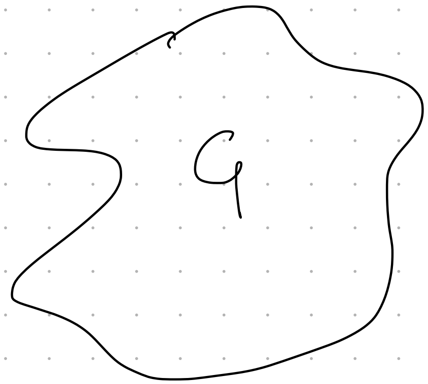
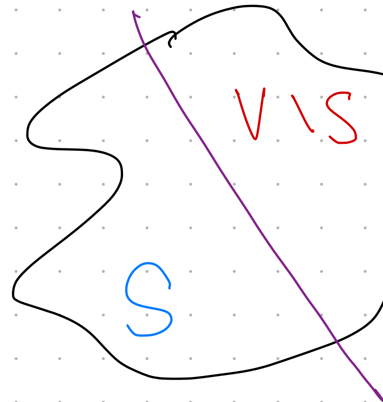


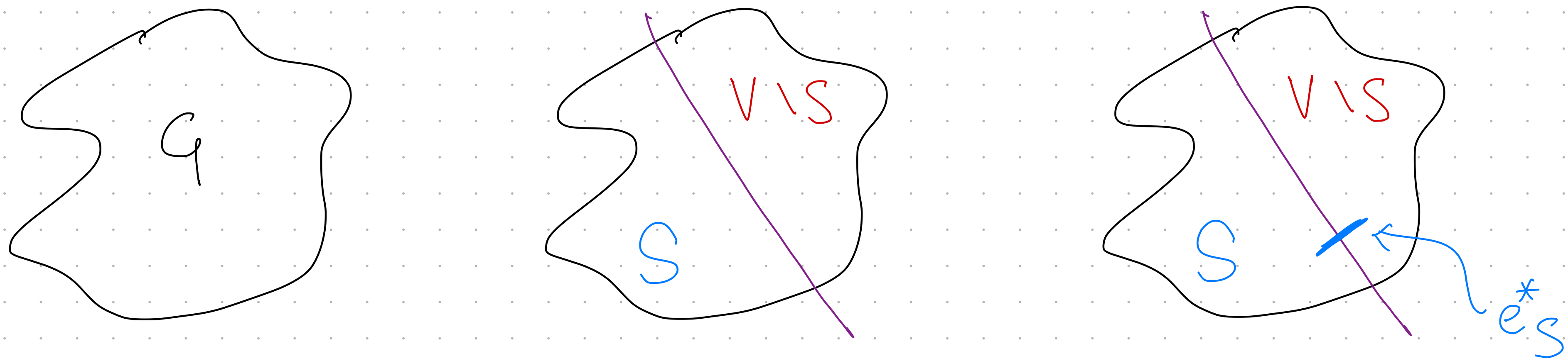


Consider *any* non-empty cut $(S, V \setminus S)$.

partition of vertices into two sets

$S \subseteq V$, $V \setminus S \subseteq V$

Cut Lemma. Given a graph, $G = (V, E, W)$.



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Denote by e_S^* the minimum-weight edge crossing the cut.

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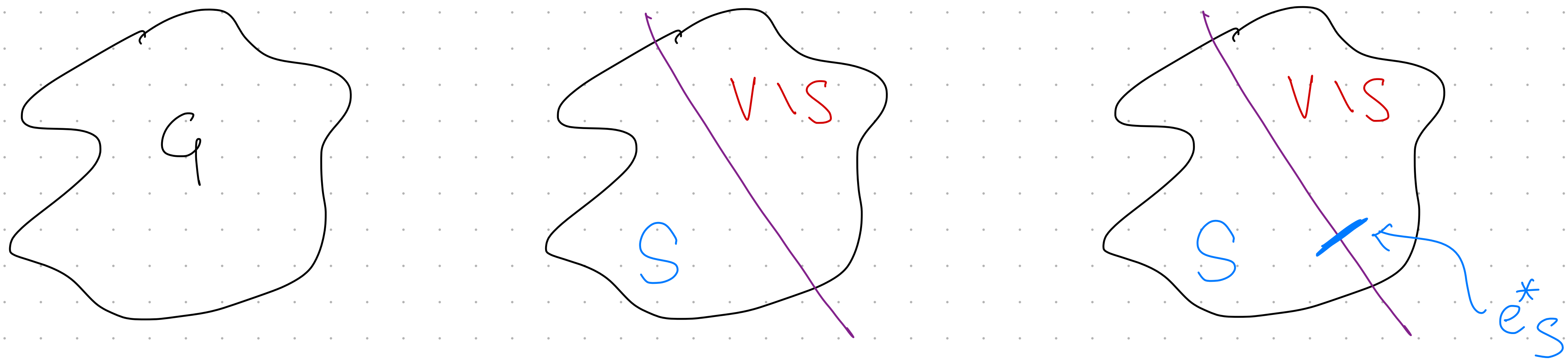

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Denote by e_S^* the minimum-weight edge crossing the cut.

$$\boxed{\text{The MST of } G \text{ contains } e_S^*.}$$

Significance of Cut Lemma?

Given $T = (V, E')$

if there exists some cut $(S, V \setminus S)$

and T does not include e_S^*, then

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T is not an MST.

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Given $T = (V, E')$

if there exists some cut $(S, V \setminus S)$

and T does not include e_S^*, then

T is not an MST.

So, any algorithm for solving MST must "collect" e_S^* for every non-empty $(S, V \setminus S)$

Proof of Cut Lemma. By exchange argument.

Start with a spanning tree T_0 that does not contain some e_S^*.

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Show how to exchange e_S^* for an edge e in T_0 s.t. the tree weight drops.

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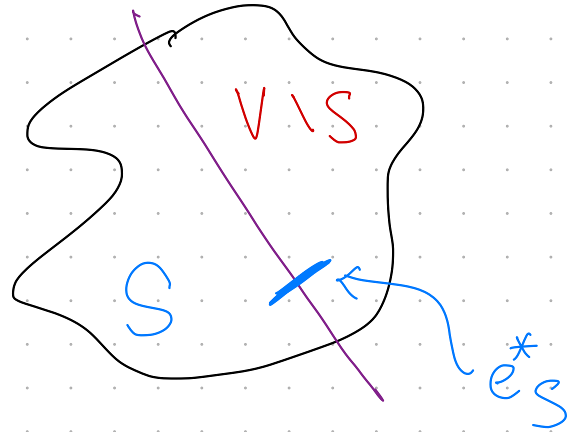
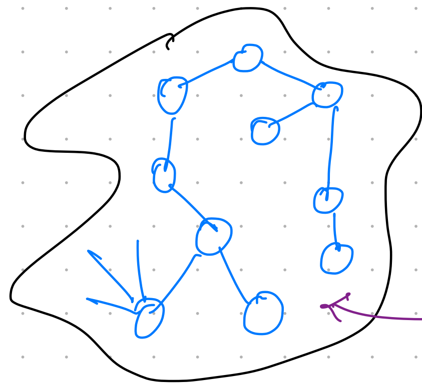
$\Longrightarrow$ T_0 cannot be MST.

$G =$



T_0



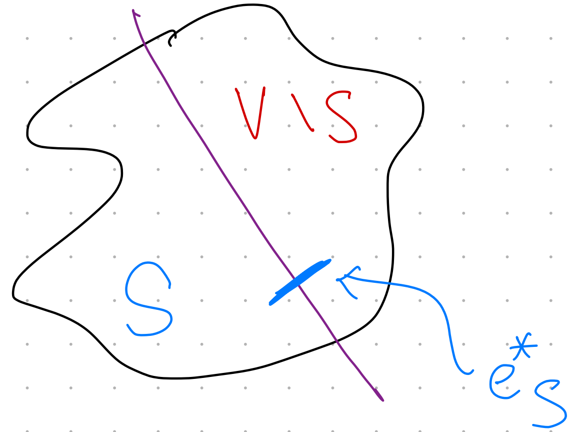
$e_S^* \notin T_0$ (by assumption)

By defn. e_S^* connects some $u \in S$, $v \in V \setminus S$

<u>Idea 0</u>. T_0 is a spanning tree $\Longrightarrow$ connected.

$\Longrightarrow$ must be some other edge from $S \longrightarrow V \setminus S$

$G =$



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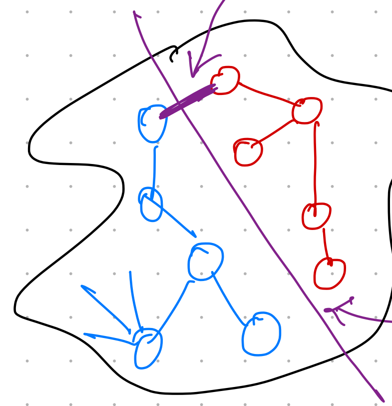


$e_0 \in T_0$ crosses $(S, V \setminus S)$

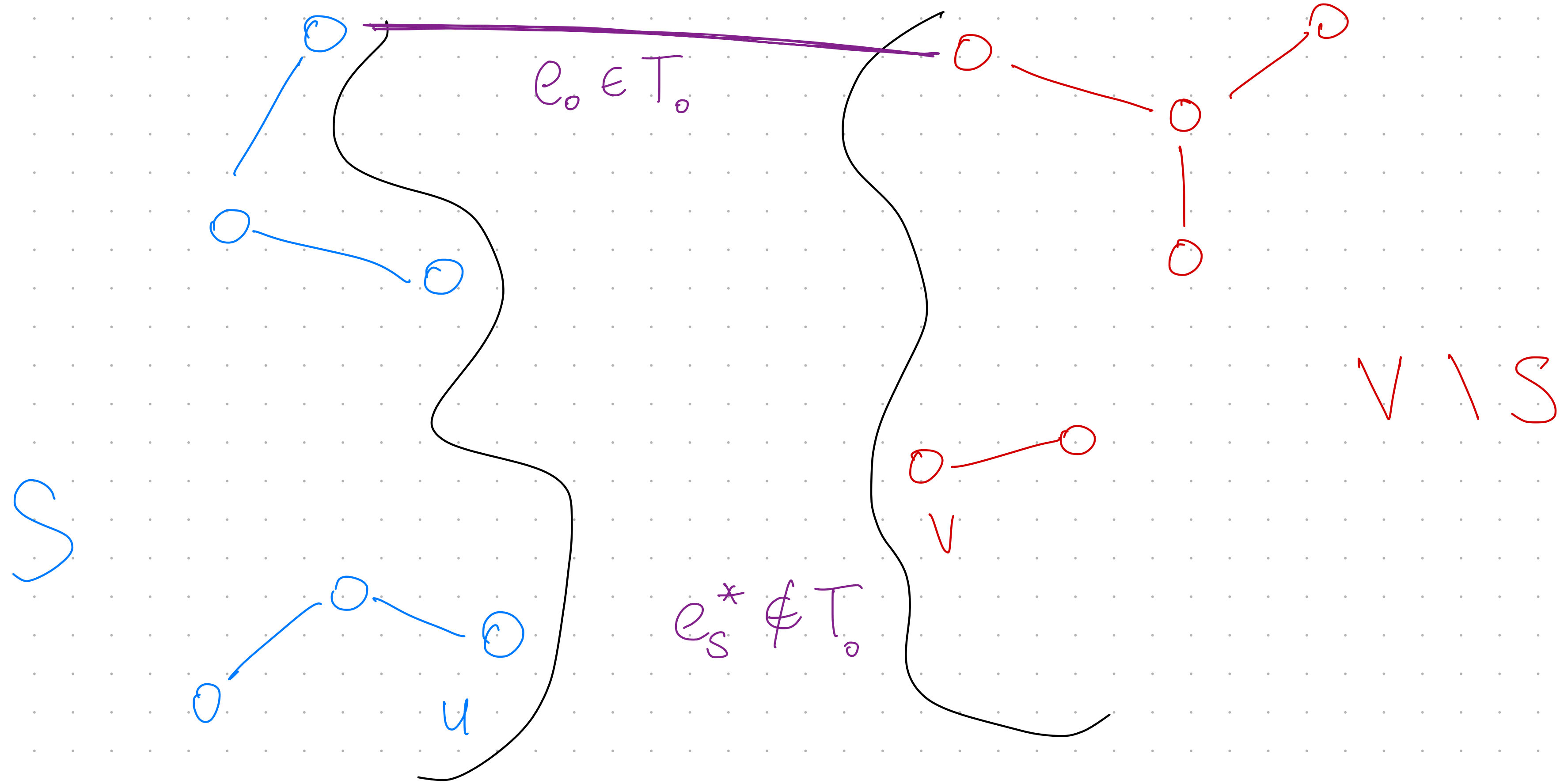
$e_S^* \notin T_0$ (by assumption)

By defn. e_S^* connects some $u \in S$, $v \in V \setminus S$

<u>Idea 0.</u> T_0 is a tree $\Longrightarrow$ connected.

$\Longrightarrow$ must be some other edge from $S \longrightarrow V \setminus S$

Idea 0. Exchange e_s^* for e_0.



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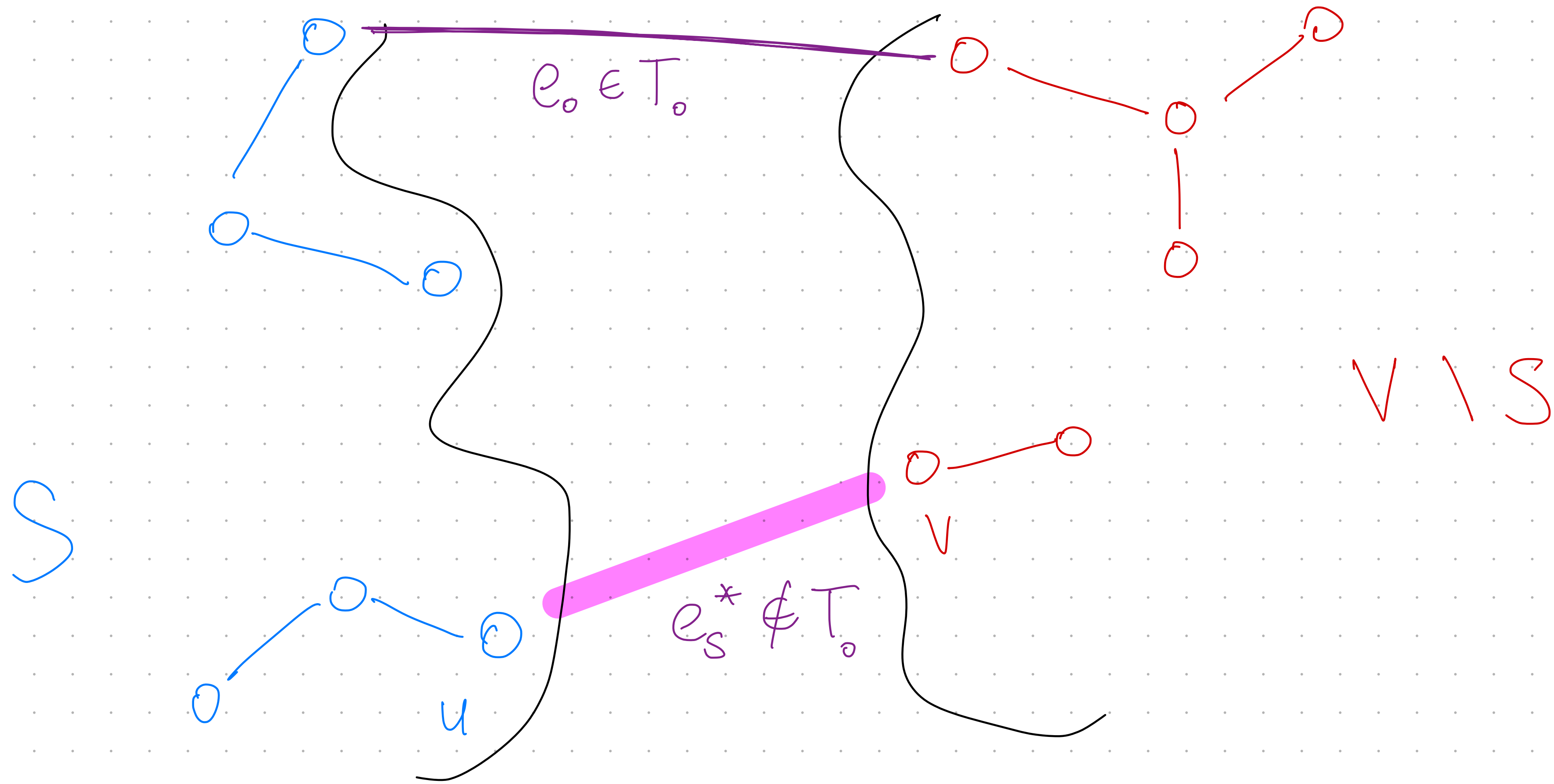

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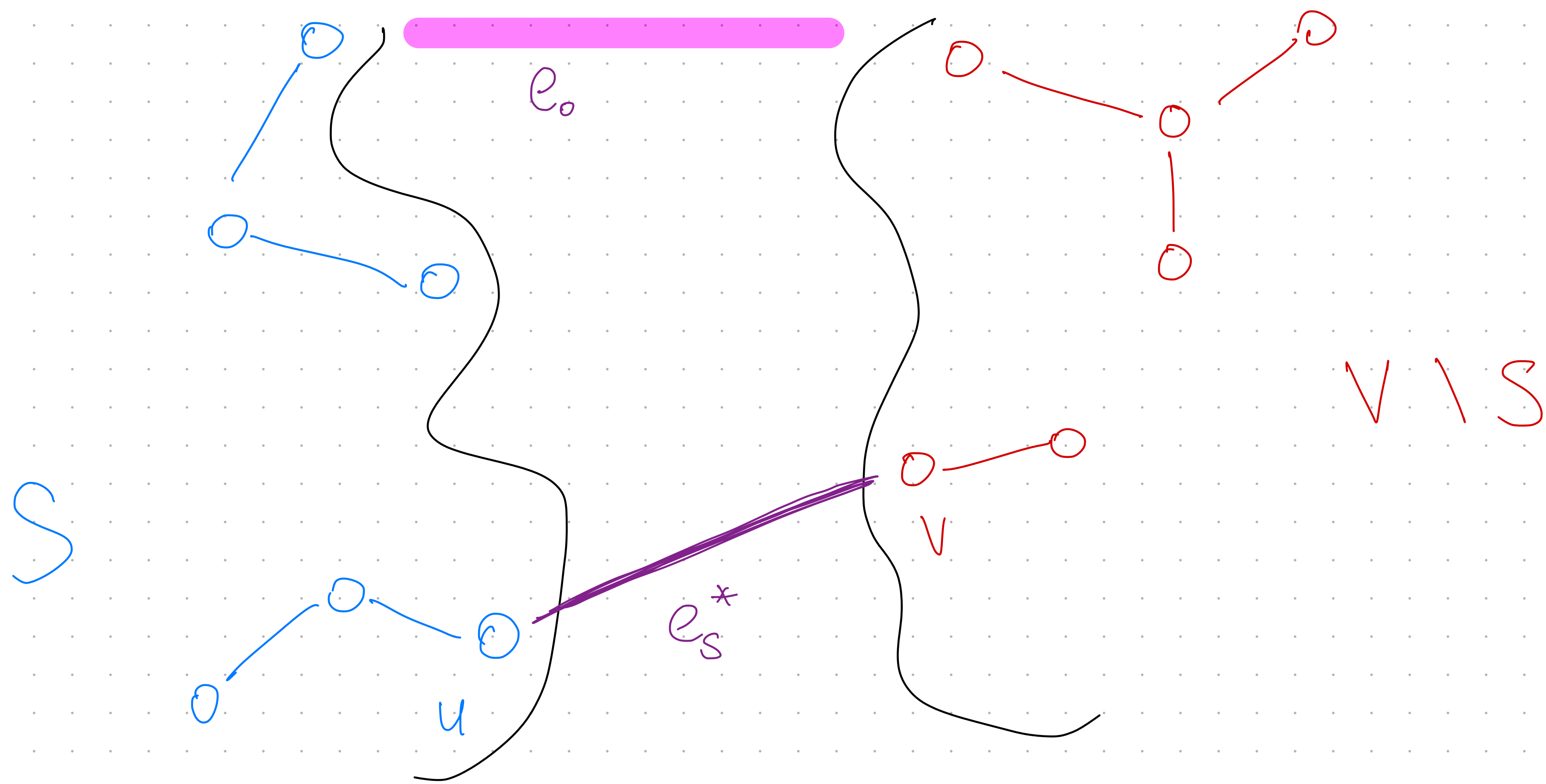
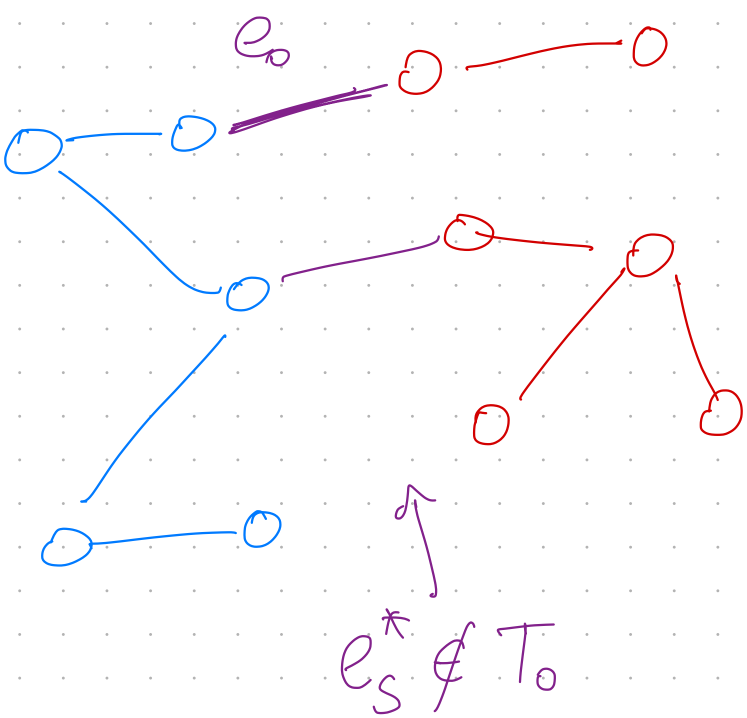


Total weight: $W_{e_s^*} - W_{e_0} + W_{T_0}$

By defn. $W_{e_s^*} < W_{e_0}$ $\Longrightarrow$ Total weight decreases!

Are we done?

No. Exchanging e_s^* for e_0 might not preserve tree structure.



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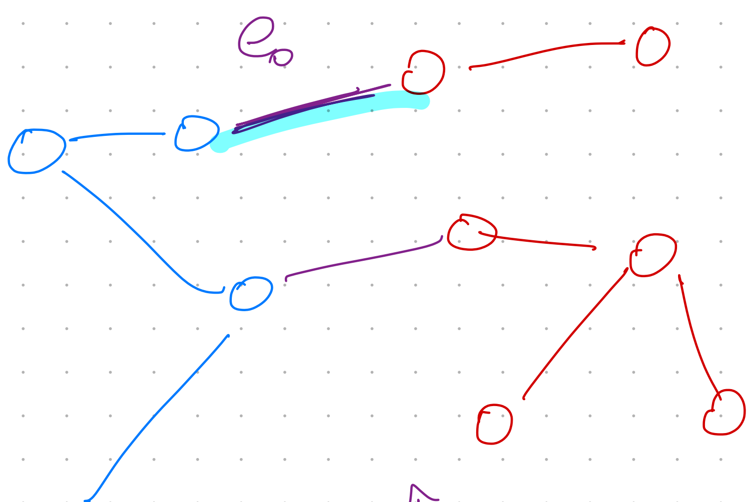

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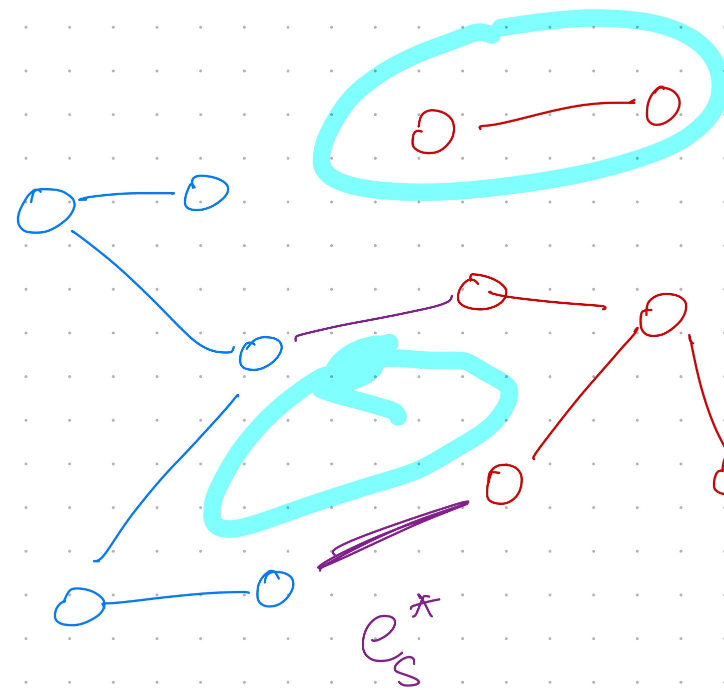
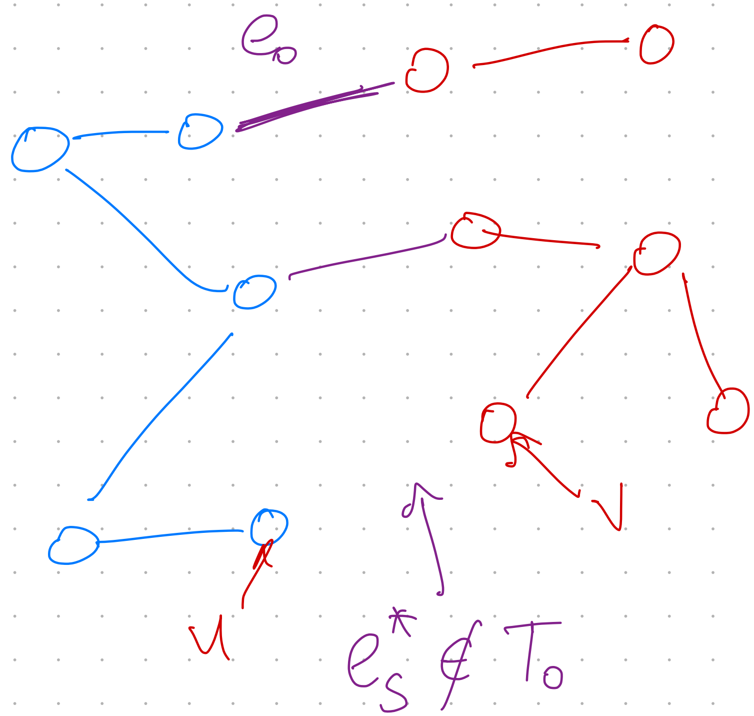


disconnected & contains cycle.

Idea 1. Find a path through T_0 connecting ends of e_s^*.



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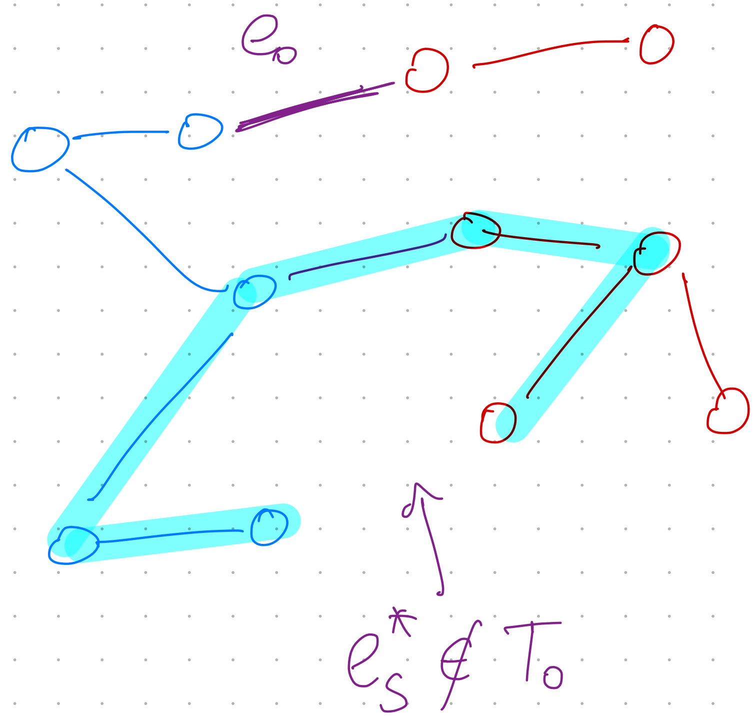

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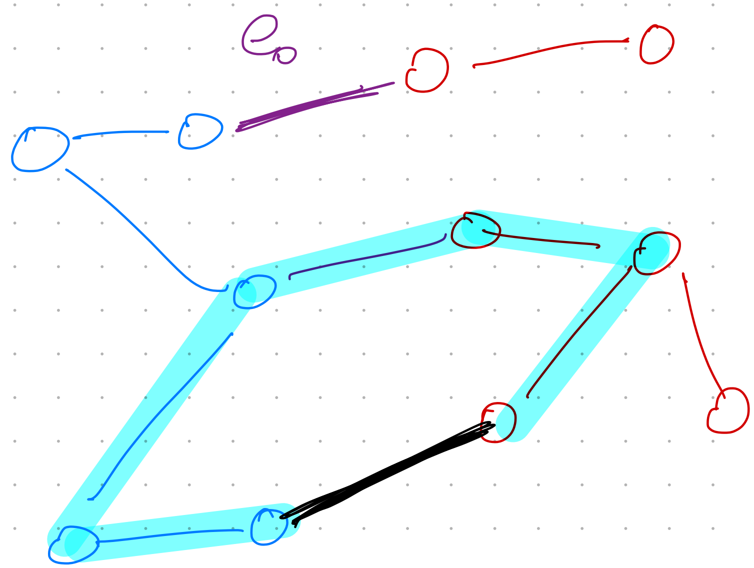

add e_s^* to form a cycle

↳ New graph still connected.

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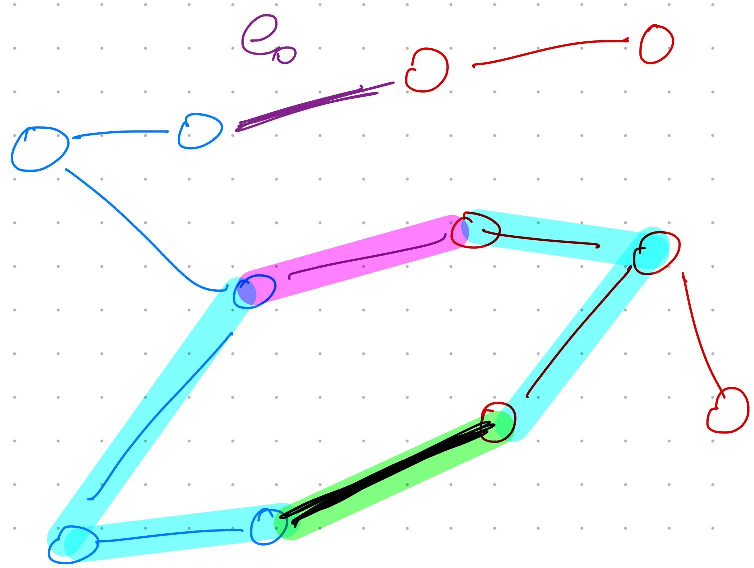

Remove edge e in the cycle, where $w_{e_s^*} < w_e$

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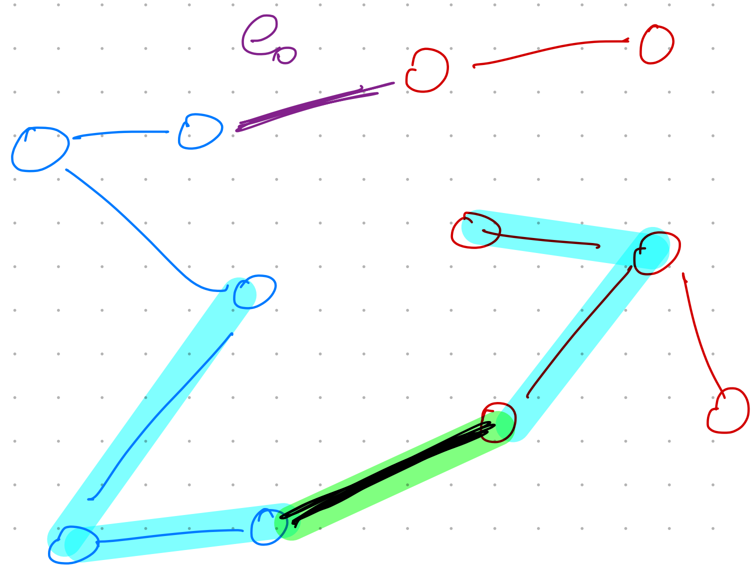

Remove edge e in the cycle, where $w_{e_s^*} < w_e$

↳ Weight goes down

↳ Removing edge from cycle cannot disconnect graph.

$\Longrightarrow$ tree $\boxtimes$