Introduction to Algorithms	**Notes in Max Cut Approximation**
CS 4820 Spring 2015	

In this note we give a simple randomized approximation algorithm for the maximum cut problem. Recall that we have shown in lecture (on March 16th) that the problem is NP-complete (see supplementary notes on the web). Section 12.4 of book gives an approximately 2-approximation using local search. In this note we will give a randomized algorithm using the notation from Section 12.4.

Consider a graph $G = (V, E)$, and let (A^*, B^*) be the maximum weight cut. Now consider the following randomized algorithm: for each node $v \in V$ decide independently with probability 1/2-1/2 which of the two sides of the cut the node is in. Let (A, B) be the resulting (random) cut. So $Pr(v \in A) = Pr(v \in B) = 1/2$ for all nodes $v \in V$, but the definition of the algorithm.

Also, we we are making the decisions on each node independently, for any two nodes u and v,

$$Pr(v, u \in A) = Pr(u \in A, v \in B) = Pr(u \in B, v \in A) = Pr(u, v \in B) = 1/4.$$

An important consequence of this fact is the following

Claim *For any edge $e = (u, v)$ the probability that the edge goes across the cut is 1/2.*

To see why, $e = (u, v)$ crosses the cut (A, B) if either $u \in A$ and $v \in B$ or if $v \in A$ and $u \in B$, which has probability 1/2.

Recall from the proof of (12.5) that if we set $W = \sum_e w_e$ to be the total weight of all edges, than clearly the maximum weight cut (A^*, B^*) satisfies $w(A^*, B^*) \leq W$. Using this fact, we can show the main result

Claim *The expected weight of the cut resulting in the algorithm above is at least $\frac{1}{2}w(A^*, B^*)$.*

Proof Let (A, B) be the random cut, and $E(w(A, B))$ the expected value of the weight of this cut. let p_e denote the probability that edge e is in this cut. But linearity of expectation $E(w(A, B)) = \sum_e p_e w_e$. We showed in the claim above that $p_e = 1/2$ for all edges e, so we get

$$E(w(A, B)) = \sum_e p_e w_e = \frac{1}{2} \sum_e w_e \geq \frac{1}{2} w(A^*, B^*),$$

as claimed.