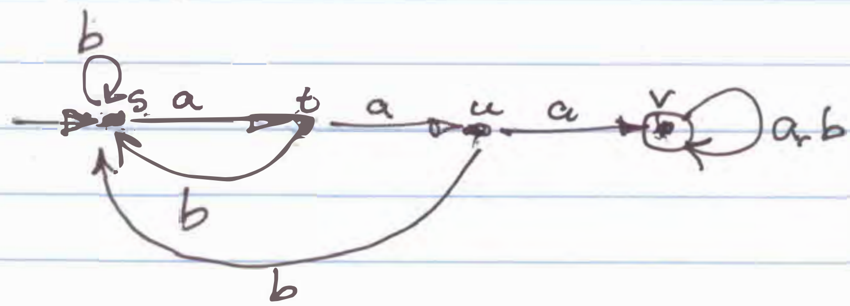
CS 4810 S22 Lecture 2 1/27/2022

DFA $\mathcal{M} = (Q, \Sigma, \delta, s, F)$

- Q: finite set ("states")
- Σ: finite set ("alphabet")
- $\delta : Q \times \Sigma \to Q$ transitions
- $s \in Q$ start state
- $F \subseteq Q$ final or accept states

Ex. 1:



$Q = \{s, t, u, v\}$ $\Sigma = \{a, b\}$ start $= s$ $F = \{v\}$

$abab\,b\underbrace{aaa}\,babb$

Extend $\delta : Q \times \Sigma \to Q$ to $\hat{\delta} : Q \times \Sigma^* \to Q$ inductively:

(A) – $\hat{\delta}(q, \varepsilon) \stackrel{\Delta}{=} q$

(B) – $\hat{\delta}(q, xa) \stackrel{\Delta}{=} \delta(\hat{\delta}(q, x), a)$, $x \in \Sigma^*$, $a \in \Sigma$.

$\hat{\delta}(q, a) = \delta(q, a)$

$$
\begin{aligned}
\hat{\delta}(q, a) &= \hat{\delta}(q, \varepsilon a) && \text{since } \varepsilon a = a \\
&= \delta(\hat{\delta}(q, \varepsilon), a) && \text{by (B)} \\
&= \delta(q, a) && \text{by (A)}.
\end{aligned}
$$

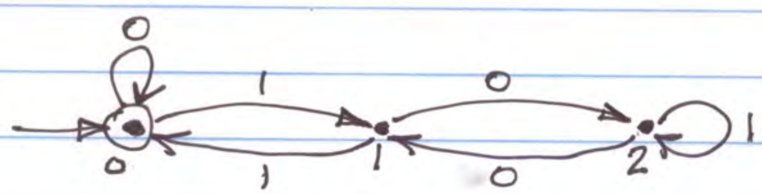
$x \in \Sigma^*$ is <u>accepted</u> by M if $\hat{\delta}(s,x) \in F$.
" <u>rejected</u> . . . . . $\hat{\delta}(s,x) \notin F$.

$$\mathcal{L}(M) \triangleq \{x \in \Sigma^* \mid \hat{\delta}(s,x) \in F\}$$

"language accepted by M"

A set $A \subseteq \Sigma^*$ is <u>regular</u> if $A = \mathcal{L}(M)$ for some DFA M.

a nonregular set: $\{a^n b^n \mid n \geq 0\}$



$\mathcal{L}(M) = \{x \in \{0,1\}^* \mid \#x \bmod 3 = 0\}$
where $\#x$ is number represented by x in binary

<u>Lemma</u>

$$\hat{\delta}(0,x) = \begin{cases} 0 & \text{if } \#x \bmod 3 = 0 \\ 1 & \text{if } \#x \bmod 3 = 1 \\ 2 & \text{if } \#x \bmod 3 = 2 \end{cases}$$

$$= \#x \bmod 3$$

$q \in \{0,1,2\}$, $c \in \{0,1\}$

~~δ(q,c)~~

	0	1
→ 0F	0	1
1	2	0
2	1	2

$$\boxed{\delta(q,c) \triangleq (2q+c) \bmod 3}$$

$\delta(2,1) = 2 = (2\cdot 2+1) \bmod 3$
$5 \bmod 3$

by convention, $\#\varepsilon = 0$

$$\left.\begin{array}{l} \#x0 = 2 \cdot \#x + 0 \\ \#x1 = 2 \cdot \#x + 1 \end{array}\right\} \quad \#xc = 2\#x + c \quad (*)$$

for $x \in \{0,1\}^*$, $c \in \{0,1\}$

$\delta(q, c) = (2q + c) \bmod 3$

<u>Lemma</u> $\hat{\delta}(0, x) = \#x \bmod 3$

- $\hat{\delta}(0, \varepsilon) = 0 = \#\varepsilon \bmod 3$ (basis)
- $\hat{\delta}(0, xc) = \delta(\hat{\delta}(0, x), c)$ by def. of $\hat{\delta}$.

$= \delta(\#x \bmod 3, c)$ by ind. hyp.

$= (2 \cdot (\#x \bmod 3) + c) \bmod 3$ by def. of δ

$= (2 \cdot \#x + c) \bmod 3$ by number thy.

$= \#xc \bmod 3$ by $(*)$

Theorem. $\mathcal{L}(\mathfrak{M}) = \{x \in \{0,1\}^* \mid \#x \bmod 3 = 0\}$

Proof. $\mathcal{L}(\mathfrak{M}) = \{x \mid \hat{\delta}(0, x) \in F\}$

$= \{x \mid \#x \bmod 3 = 0\}$

Closure properties

If A, B are regular, then so are

① $A \cup B$

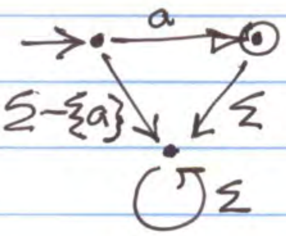
② $A \cap B$

③ $\sim A$

④ $AB = \{xy \mid x \in A \ \& \ y \in B\}$

⑤ $A^* = \bigcup_{n \geq 0} A^n$

Regular sets are smallest family of sets containing $\{a\}$ for $a \in \Sigma$ & closed under ①-⑤.



sets over Σ^*

Theorem If A, B are regular, then so is $A \cap B$.

$\subseteq \Sigma^*$

Proof. Product construction.

$A = \mathcal{L}(M_1)$, $B = \mathcal{L}(M_2)$

$$M_1 = (Q_1, \Sigma, \delta_1, s_1, F_1)$$

$$M_2 = (Q_2, \Sigma, \delta_2, s_2, F_2)$$

Define $M_3 = (Q_3, \Sigma, \delta_3, s_3, F_3)$

where:

$$Q_3 \triangleq Q_1 \times Q_2 = \{(q_1, q_2) \mid q_1 \in Q_1 \ \& \ q_2 \in Q_2\}$$

$$s_3 \triangleq (s_1, s_2) \qquad F_3 \triangleq F_1 \times F_2$$

$$\delta_3 : Q_3 \times \Sigma \to Q_3$$

$$\delta_3((q_1, q_2), a) \triangleq (\delta_1(q_1, a), \delta_2(q_2, a))$$

Lemma. $\hat{\delta}_3((q_1, q_2), x) = (\hat{\delta}_1(q_1, x), \hat{\delta}_2(q_2, x))$

Proof. By ind. on $|x|$.

Basis: $x = \varepsilon$:

$\hat{\delta}_3((q_1, q_2), \varepsilon) = (q_1, q_2)$ by def. of $\hat{\delta}_3$

$= (\hat{\delta}_1(q_1, \varepsilon), \hat{\delta}_2(q_2, \varepsilon))$ by def. of $\hat{\delta}_1, \hat{\delta}_2$

Ind. step: xa

$\hat{\delta}_3((q_1, q_2), xa) = \delta_3(\hat{\delta}_3((q_1, q_2), x), a)$ def of $\hat{\delta}_3$

$= \delta_3((\hat{\delta}_1(q_1, x), \hat{\delta}_2(q_2, x)), a)$ by I.H.

$= (\delta_1(\hat{\delta}_1(q_1, x), a), \delta_2(\hat{\delta}_2(q_2, x), a))$
by def of δ_3

$= (\hat{\delta}_1(q_1, xa), \hat{\delta}_2(q_2, xa))$ def. of $\hat{\delta}_1, \hat{\delta}_2$

Theorem $\mathcal{L}(\mathfrak{M}_3) = \mathcal{L}(\mathfrak{M}_1) \cap \mathcal{L}(\mathfrak{M}_2)$

$x \in \mathcal{L}(\mathfrak{M}_3) \Leftrightarrow \hat{\delta}_3(s_3, x) \in F_3$ def. of acceptance

$\Leftrightarrow \hat{\delta}_3((s_1, s_2), x) \in F_1 \times F_2$ def of s_3, F_3

$\Leftrightarrow (\hat{\delta}_1(s_1, x), \hat{\delta}_2(s_2, x)) \in F_1 \times F_2$ by Lemma

$\Leftrightarrow \hat{\delta}_1(s_1, x) \in F_1$ & $\hat{\delta}_2(s_2, x) \in F_2$ by def of $F_1 \times F_2$

$\Leftrightarrow x \in \mathcal{L}(\mathfrak{M}_1)$ & $x \in \mathcal{L}(\mathfrak{M}_2)$ def of acceptance

$\Leftrightarrow x \in \mathcal{L}(\mathfrak{M}_1) \cap \mathcal{L}(\mathfrak{M}_2) = A \cap B$ def of $\cap$

Since true for all x, $\mathcal{L}(\mathfrak{M}_3) = A \cap B$.

If A, B are regular, so are $\sim A$, $A \cap B$,

$A \cup B = \sim(\sim A \cap \sim B)$ by de Morgan law.

$$M_3' = (Q_3, \Sigma, \delta_3, s_3, F_3')$$

$$F_3' = (Q_1 \times F_2) \cup (F_1 \times Q_2)$$

AB, A^* later!