Notes from after-class discussion on homomorphisms -
these are pretty messy, sorry :(

$c \in A$ constant

$h: A^n \to A$ n-ary $\cancel{c: A^0 \to A}$ = 0-ary

$f: A^2 \to A$ binary $g: A \to A$ unary

<u>Signature</u>: function symbols with <u>arities</u>

$(\cancel{A, f}$

<u>Ex</u>. Group signatures: $\cdot : A^2 \to A$ $(\cdot)^{-1}: A \to A$

$1 : A^0 \to A$

$(A, \cdot, ^{-1}, 1)$

Rings: $(A, +, \cdot, 0, 1, -)$

$(\mathbb{Z}, +, \cdot, 0, 1, -)$

Monoids: $(A, \cdot, 1)$ $\cdot : A^2 \to A$ $1 : A^0 \to A$

---

homomorphism (A, f^A, g^A, c^A) f: binary g: unary c: constant

(B, f^B, g^B, c^B)

$h: A \to B$

$\forall x_1, x_2 \in A$ $\quad h(\overbrace{f^A(x_1, x_2)}^{h(x_1 x_2)}) = f^B(h(x_1), h(x_2))$ $\quad h(x_1) \cdot h(x_2)$

$\forall x \in A$ $\quad h(g^A(x)) = g^B(h(x))$

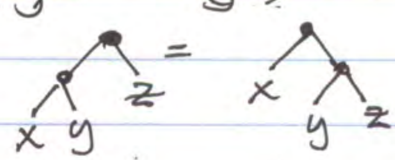
$h(c^A) = c^B$

$\Sigma = \{a, b\}$ $(\Sigma^*, \cdot, \varepsilon)$ is a monoid.

$\Gamma = \{c, d\}$ $(\Gamma^*, \cdot, \varepsilon)$ $\quad \therefore xyz$

$h(a) = cd$ $\quad (xy)z = x(yz)$

$h(b) = ddc$



$h(ab) = cdddc$

$h(ab) = h(a) \cdot h(b)$

$h(a_1 a_2 \cdots a_n) = h(a_1) \cdots h(a_n)$ $\quad \varepsilon x = x \varepsilon = x$

$h(\varepsilon) = \varepsilon$

$(\Sigma^*, \cdot, \varepsilon) \quad xy \neq yx$

$(\mathbb{N}, +, 0) \qquad (x+y)+z = x+(y+z)$

$x+y = y+x \qquad 0+x = x+0 = 0$

$|\cdot| : \Sigma^* \to \mathbb{N}$ is monoid homomorphism.

$|\cdot| : (\Sigma^*, \cdot, \varepsilon) \longrightarrow (\mathbb{N}, +, 0)$

$|xy| = |x| + |y|$

$|\varepsilon| = 0$

$(xy)z = x(yz) \qquad \cancel{\varepsilon x = x\varepsilon = x}$

$(M, \cdot, 1) \qquad x \cdot 1 = 1 \cdot x = x$

$(A, +, \cdot, 0, 1, {*})$
$\;\;\; 2 \;\; 2 \;\; 0 \;\; 0 \;\; 1$

$A \cup B$

$A \cdot B$

$A^* = \bigcup_{n \geq 0} A^n$

$(2^{\Sigma^*}, \cup, \cdot, \emptyset, \{\varepsilon\}, {*})$
Kleene algebra.

$\{a\} \subseteq \Sigma^*$

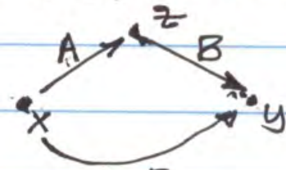
$\cup\!\!\mid$

$(\mathrm{Reg}\,\Sigma, \cup, \cdot, \emptyset, \{\varepsilon\}, {*})$ 

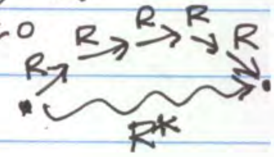
Kleene algebras.

$A, B \subseteq X \times X = \{(x,y) \mid x, y \in X\}$

$A \circ B = \{(x,y) \mid \exists z\; (x,z) \in A,\ (z,y) \in B\}$



$R^* = \bigcup_{n \geq 0} R^n$



$(2^{X \times X}, \cup, \circ, \emptyset, \{(x,x) \mid x \in X\}, {*})$
is a K.A.