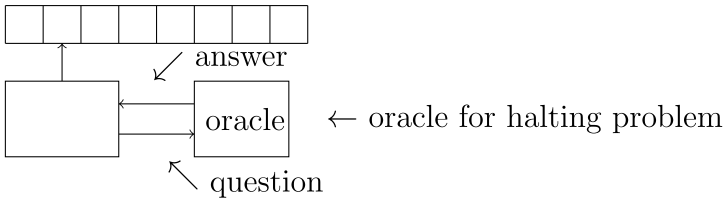
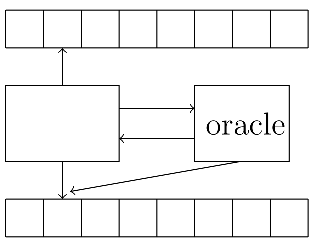
Oracles

What if one could solve the halting problem?

New kind of Turing machine



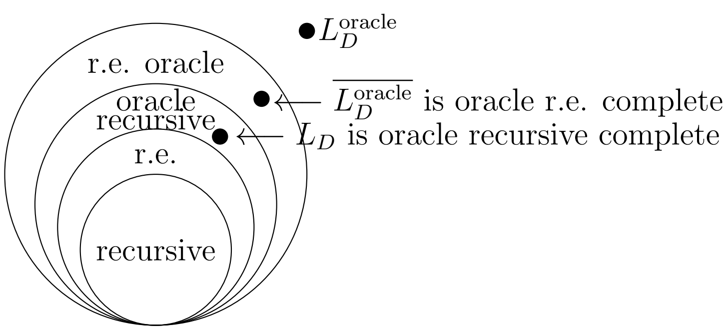
How do I ask a question? Turing machine writes question on second tape and oracle reads question and gives yes or no answer.



2^{nd} tape for questions to oracle.

The halting oracle answers the question "Does M_i on x_j halt?"
Let $S = \{(M, x)|M$ halts on $x\}$

Just as we defined recursive and r.e. we can define oracle recursive and oracle r.e. with respect to the oracle S.



1. (a) r.e. sets are oracle recursive The oracle can tell if a Tm halts.
   (b) $L_D = \{x_i|x_i \notin L(M_i)\}$ oracle recursive since the oracle can solve the halting problem. L_D is oracle recursive hard since $\{(M, x)|x \in L(M)\}$ is polynomial time reducible to L_D. Create M_x that accepts input M_x if $x \in L(M)$. Use L_D to determine if $M_x \in L(M)$.

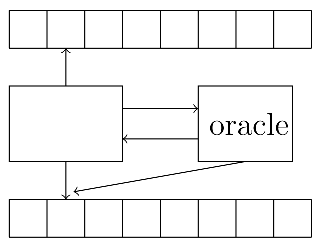
2. ∃ oracle r.e. sets not oracle recursive. The sets of oracle Tm's can be listed and we can by diagonalization define L_D^{oracle}.

There is a hierarchy of undecidable problems. Let M be a regular Turing machine.

1. Is $x \epsilon L(M)$? is r.e.
2. Does $L(M) = \phi$? equivalent to does there exists a valid computation.
3. Is $L(M)$ infinite?
4. Is $L(M)$ a regular set?
5. Is $L(M)$ a specific infinite set? $\forall x$ in set ∃ valid comp
6. Is $L(M)$ a regular set? ∃ set $\forall x$ in set ∃ valid comp.

Oracles

Attach an oracle for the halting problem to a Turing machine.



2^{nd} tape for questions to oracle.

Define M^S to be a Turing machine with an oracle for the set S.
Define $S_1 = \{M | L(M) = \Phi\}$.
The set S_1 is not r.e. The complement set $\{M | L(M) \neq \Phi\}$ is r.e.
Define $S_i = \{M^{S_{i-1}} | L(M^{S_{i-1}}) = \Phi\}$.

The set $\{(M, x) | M$ halts when started on $x\}$ is equivalent to the complement of S_1. Equivalent means same complexity: the two sets are reducible to each other by an algorithm that halts on all inputs.

Reductions

$\{(M, x) | M$ halts on $x\}$ reduction to $\{M | L(M) \neq \Phi\}$

To determine if M halts on x create M_x that on every input simulates M on x and if M halts then M_x accepts its input. Thus

$$L(M_x) = \begin{cases} \Sigma^* & M \text{ halts on } x \\ \Phi & \text{otherwise} \end{cases}$$

$\{M|L(M) \neq \Phi\}$ reduction to $\{(M,x)|M$ halts on $x\}$

> To determine if $L(M) \neq \Phi$ reduction to "M_x halts on x" create M_x that on input x starts simulating M on longer and longer inputs for more and more steps looking for a string x that M accepts. Use the (i,j) technique where one simulates the i^{th} Turing machine for j steps. If M accepts some string then M_x halts and accepts x. Otherwise M_x runs forever.

The set $\{M|L(M) = \Sigma^*\}$ is harder than membership. It is equivalent to $S_2 = \{M^{S_1}|L(M^{S_1}) = \Phi\}$.

Reduction of $\{M|L(M) = \Sigma^*\}$ to $S_2 = \{M^{S_1}|L(M^{S_1}) = \Phi\}$.

> Design M^{S_1} so that
>
> $$L(M^{S_1}) = \begin{cases} \Phi & \text{if } L(M) = \Sigma^* \\ \Sigma^* & \text{otherwise} \end{cases}$$
>
> If we could determine if $L(M^{S_1}) = \Phi$. then we could determine if $L(M) = \Sigma^*$ M^{S_1} on every input looks for x not in $L(M)$. If M^{S_1} finds an x not in $L(M)$ then M^{S_1} accepts its input. To find x not in $L(M)$ requires the halting problem.

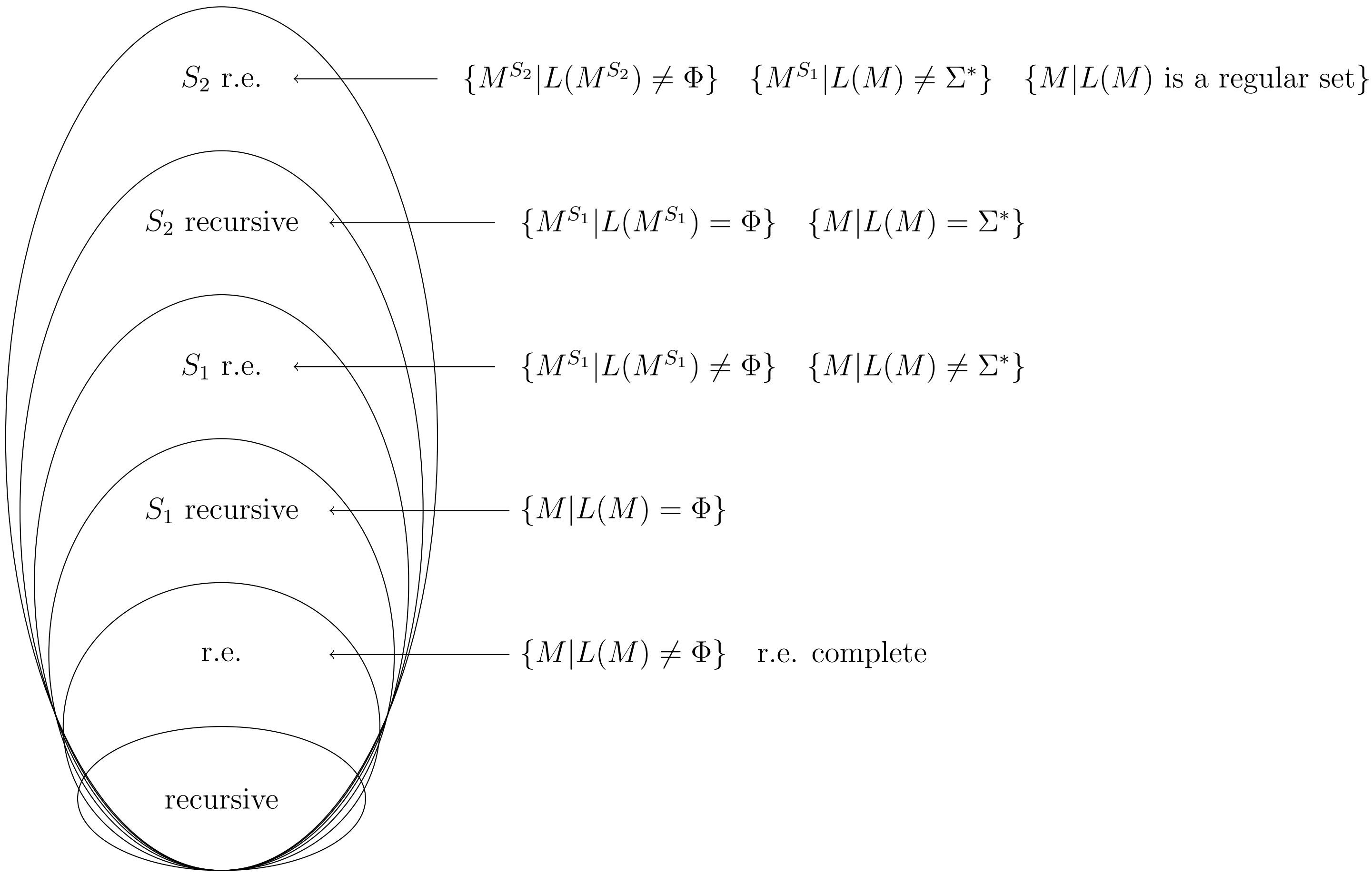
Reduction of $S_2 = \{M^{S_1}|L(M^{S_1}) = \Phi\}$ to $\{M|L(M) = \Sigma^*\}$. To determine if $L(M^{S_1}) = \Phi$ create M that accepts all invalid computations of M^{S_1}. If the set of invalid computations is Σ^* then $L(M^{S_1}) = \Phi$.

> A valid computation of M^{S_1} has $q?$ inserted when M^{S_1} asks a question followed by a valid computation of an ordinary Turing machine followed by q_y if the answer is yes and just the symbol q_n if the answer is no since one needs an oracle when the answer is no. The only complicated issue in checking that the string is an invalid computation is when the oracle returns a no.
> In this case the oracle has said that $L(M) = \Phi$ when actually accepting some string. One can determine this by the (i,j) technique of simulating M on x_i for j steps. Use oracle to determine if process halts.

The set $\{M|L(M)$ is a regular set$\}$ is equivalent to S_3.

A hierarchy

$L(M) \neq \Phi$	$\exists x\ x \in L(M)$	$\exists$ valid comp
$L(M) = \Phi$	$\forall x\ x \notin L(M)$	$\forall$ strings not valid comps
$L(M) \neq \Sigma^*$	$\exists\ x\ \forall$ comps	comp not valid
$L(M) = \Sigma^*$	$\forall x\ \exists$ valid comp	
$L(M)$ regular set	$\exists$ regular set $\forall x\ x$ in R iff x in $L(M)$	



$$S_1 = \{M|L(M) = \Phi\} \qquad S_2 = \{M^{S_1}|L(S^{S_1}) = \Phi\}$$

1. (a) $\{M|L(M) \neq \Phi\}$ is r.e. Run M on longer and longer inputs for more and more steps looking for x that is accepted.

   (b) $\{M|L(M) \neq \Phi\}$ is complete for r.e. One can answer does M accept x for any Turing machine M. Create M_x where $L(M_x) = \begin{cases} \Sigma^* & \text{if } M \text{ accepts } x \\ 0 & \text{otherwise} \end{cases}$

2. (a) $\{M|L(M) = \Phi\}$ is S_1 recursive Use oracle to determine whether or not M in S_1.

   (b) $\{M|L(M) = \Phi\}$ is S_1 complete

3. (a) $\{M^{S_1}|L(M^{S_1}) \neq \Phi\}$ is S_1 r.e. Run M^{S_1} on longer and longer inputs for more and more steps looking for x that is accepted.

   (b) $\{M|L(M) \neq \Sigma^*\}$ is S_1 r.e. Use oracle for $\{[M,x]|x \in L(M)\}$ which is equivalent to S_1 to search for $x \neq L(M)$. If it finds $x \notin L(M)$ it halts. Otherwise it runs forever.

4. (a) $\{M^{S_1}|L(M^{S_1}) = \Phi\}$ is S_2 recursive   Use oracle for S_1 to determine whether or not M^{S_1} in S_1.

   (b) $\{M|L(M) = \Sigma^*\}$ is S_2 recursive   Create M^{S_1} that searches for $x \notin L(M)$. The S_1 oracle answers if a given x is or is not in $L(M)$. If M^{S_1} finds an x not in $L(M)$ it stops, otherwise it runs forever. Use S_2 to determine if M^{S_1} stops.

5. (a) $\{M^{S_2}|L(M^{S_2}) \neq \Phi\}$ is S_2 r.e.   Run M^{S_2} on longer and longer inputs for more and more steps looking for x that is accepted.

   (b) $\{M^{S_1}|L(M^{S_1}) \neq \Sigma^*\}$ is S_2 r.e.   Look at longer and longer x using S_2 to determine if $x \in L(M^{S_1})$. If we find string halt and say yes. Otherwise run forever.

   (c) $\{M|L(M)$ is a regular set$\}$ is S_2 r.e.
   Cycle through all regular sets.
   For each regular set R ask if $L(M) = R$.
   To do this cycle through all x
   ask is x in $L(M)$ using S_1.
   ask if x is found proving $L(M) \neq R$ by S_2. If yes move on to next R otherwise return yes $L(M)$ is regular.
   If regular set found we return yes. Else we run forever. Thus S_2 r.e.