Spectral Embedding

Consider an undirected graph $G = (V, E)$ consisting of $|V| = n$ vertices and E is the set of edges that indicate which pairs of vertices are connected. Let A be the $n \times n$ adjacency matrix of the graph where $A[i, j] = 1$ if node i and node j have an edge between them. Let D be the $n \times n$ diagonal matrix with entry $D[i, i]$ indicating the degree of node i in the graph.

Question 1: Let y be any n dimensional vector (one coordinate for every node in the graph). Show that

$$\sum_{(u,v)\in E} (\mathbf{y}_u - \mathbf{y}_v)^2 = 2\ \mathbf{y}^\top (D - A)\mathbf{y}$$

Question 2: This is a simple one given what we showed in question 1. Define the matrix $L = D - A$ (this is called the Laplacian matrix)

1. **Show that L is positive semidefinite.**
2. **If we take 1 to be the all ones vector, show that $L\ \mathbf{1} = 0$**