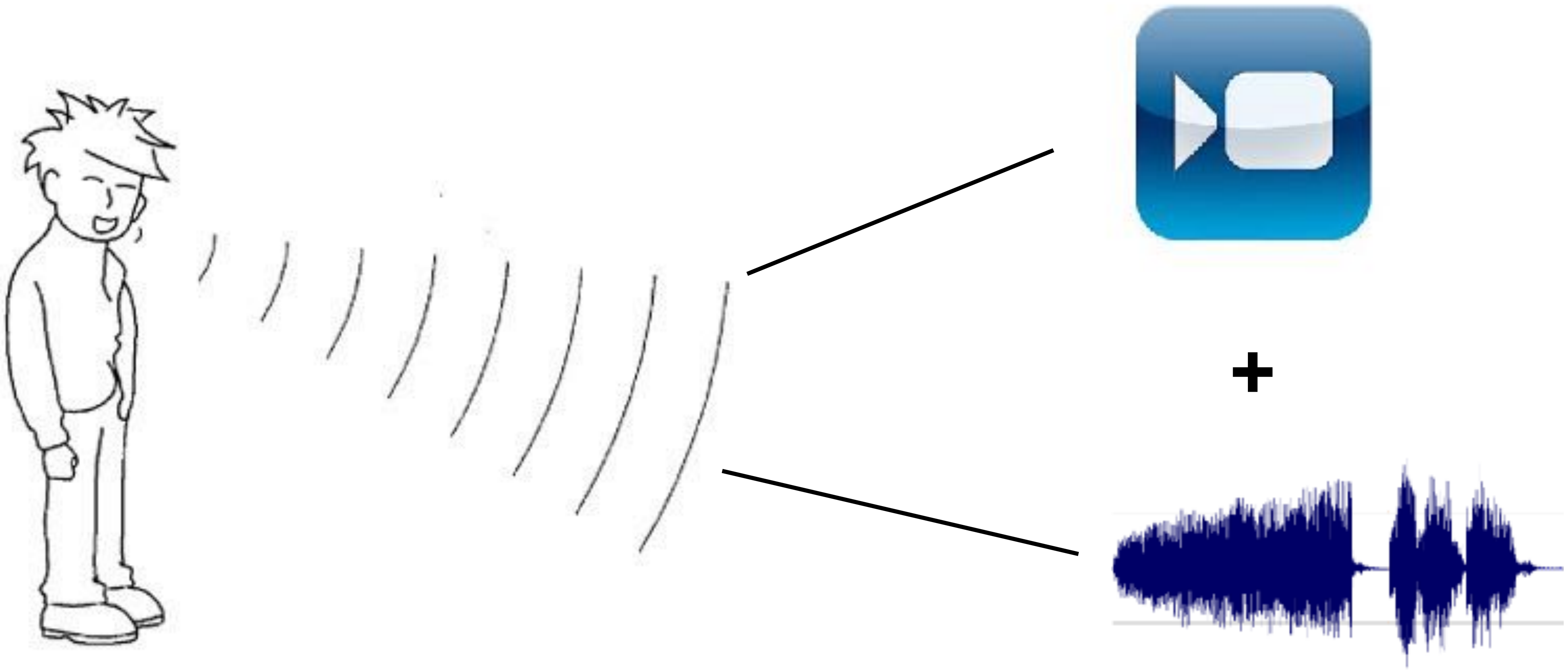
Machine Learning for Data Science (CS4786)
Lecture 6

Canonical Correlation Analysis

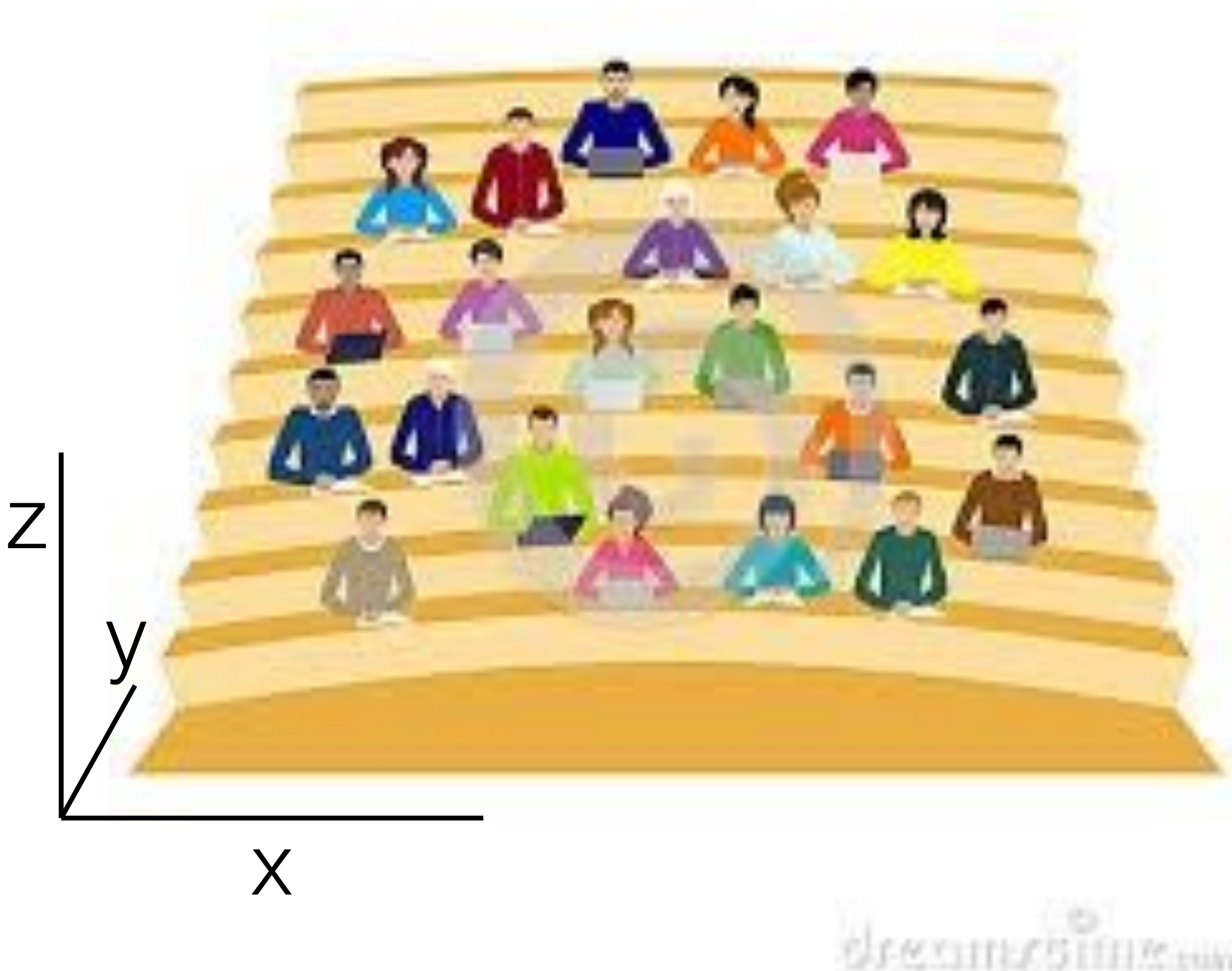
Example I: Speech Recognition



- Audio might have background sounds uncorrelated with video
- Video might have lighting changes uncorrelated with audio
- Redundant information between two views: the speech

Canonical Correlation Analysis

Canonical Correlation Analysis



Age

\+ Gender

Candies per week

Favorite Cartoon

Two View Dimensionality Reduction

- Data comes in pairs $(\mathbf{x}_1, \mathbf{x}'_1), \ldots, (\mathbf{x}_n, \mathbf{x}'_n)$ where $\mathbf{x}_t$'s are d dimensional and $\mathbf{x}'_t$'s are d' dimensional

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- Goal: Compress say view one into $\mathbf{y}_1, \ldots, \mathbf{y}_n$, that are K dimensional vectors
  - Retain information redundant between the two views
  - Eliminate "noise" specific to only one of the views

Example II: Combining Feature Extractions

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- Method A and Method B are both equally good feature extraction techniques
- Concatenating the two features blindly yields large dimensional feature vector with redundancy
- Applying techniques like CCA extracts the key information between the two methods
- Removes extra unwanted information

How do we get the right direction? (single dimension K =1)

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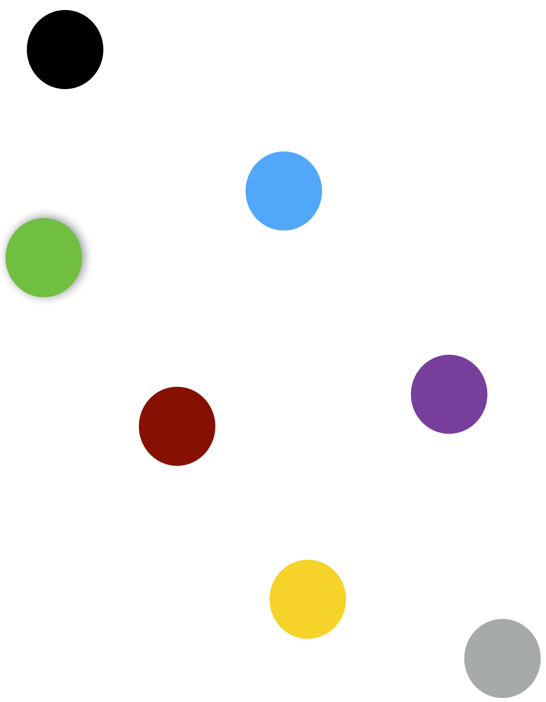
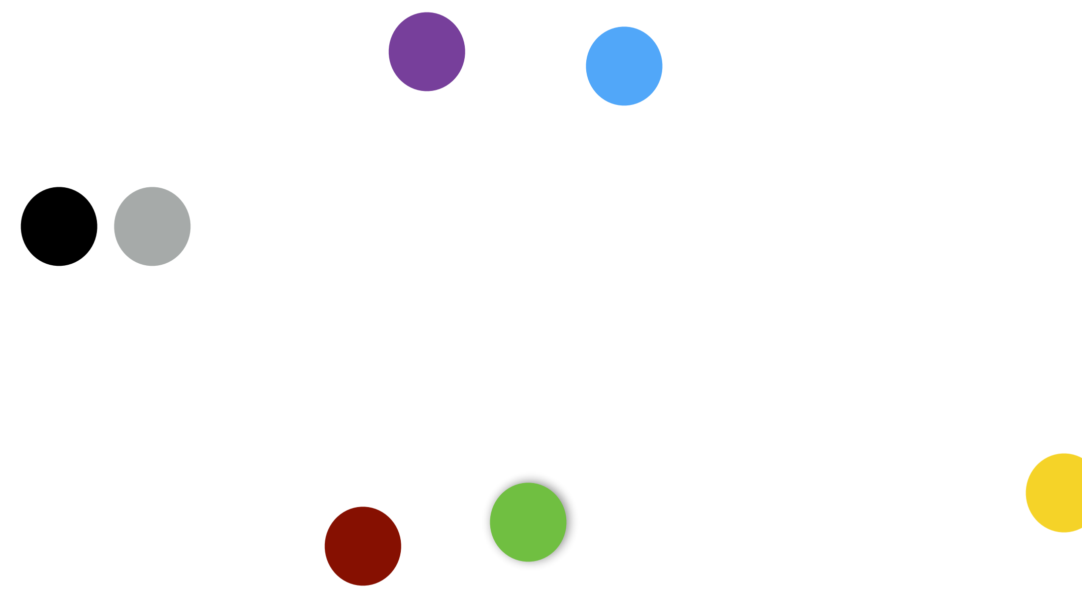
Which Direction to Pick?

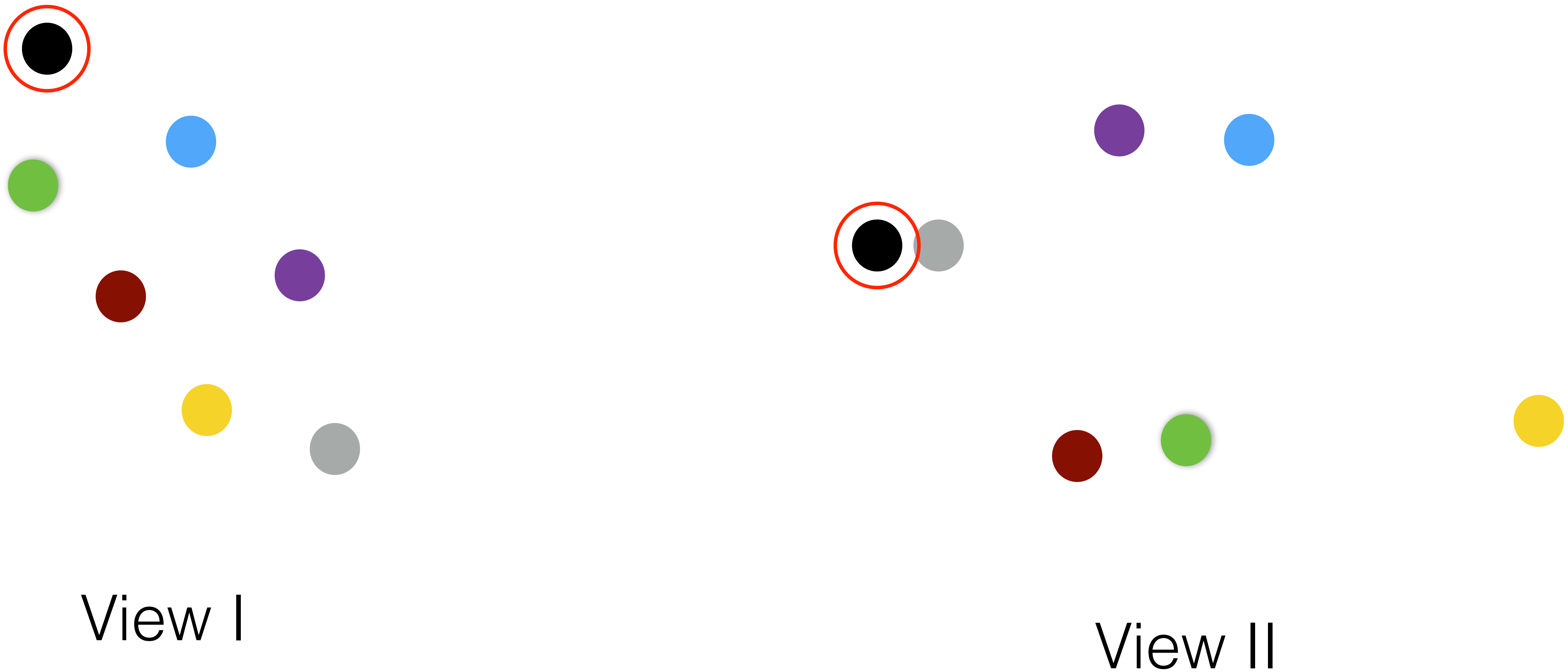


View I



View II

Which Direction to Pick?



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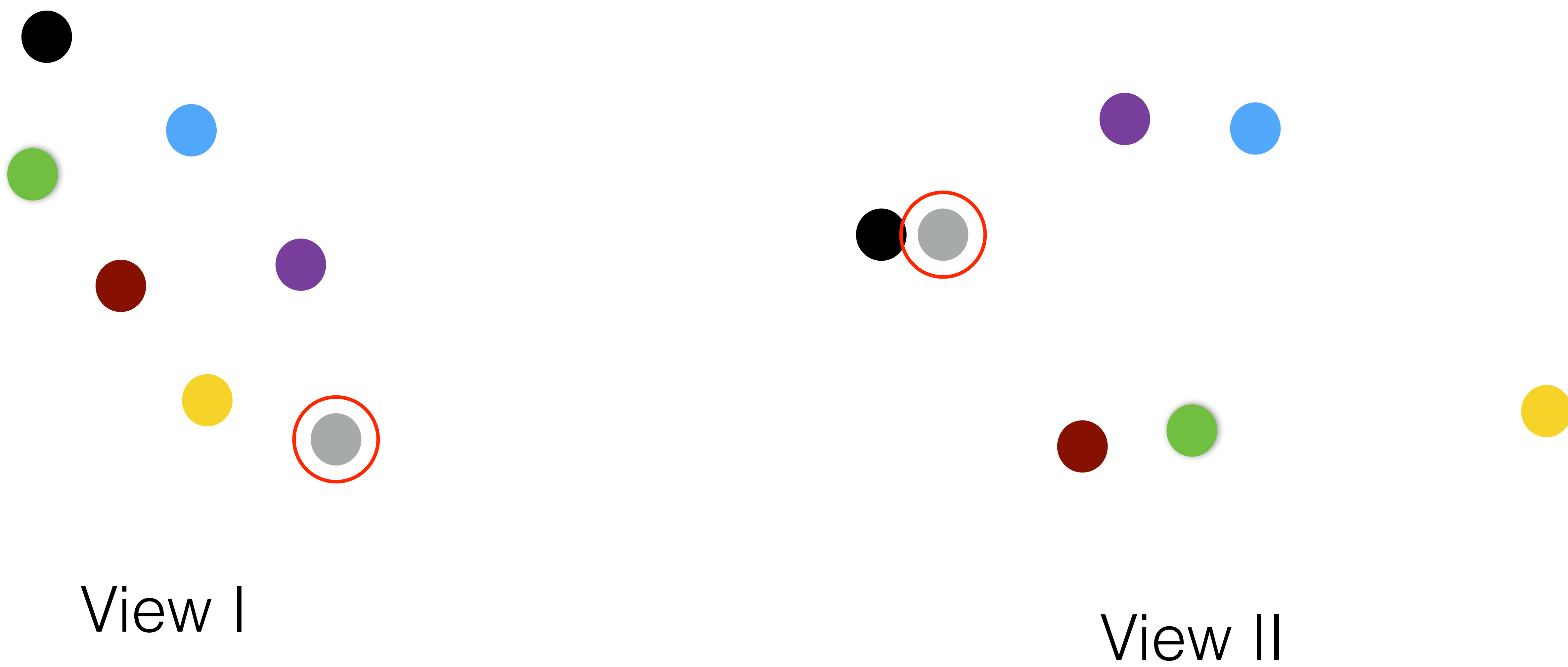

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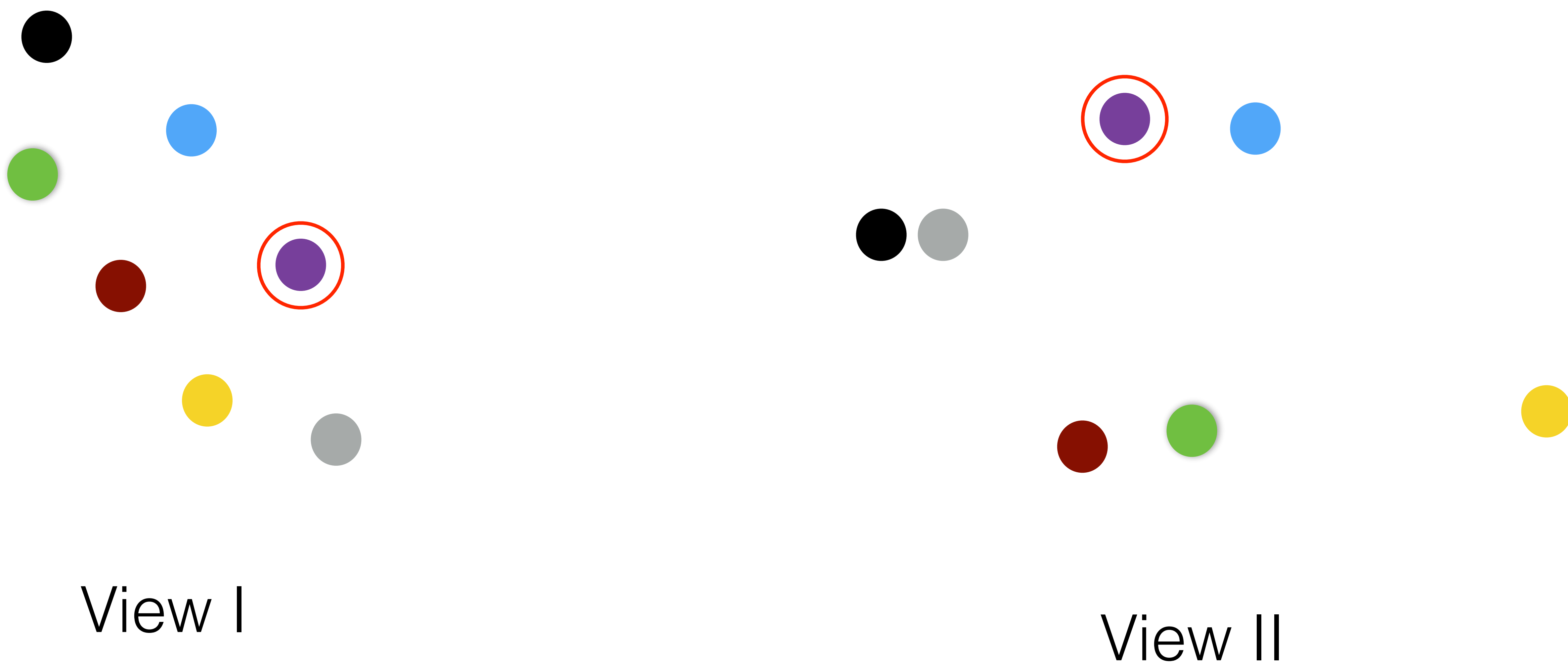

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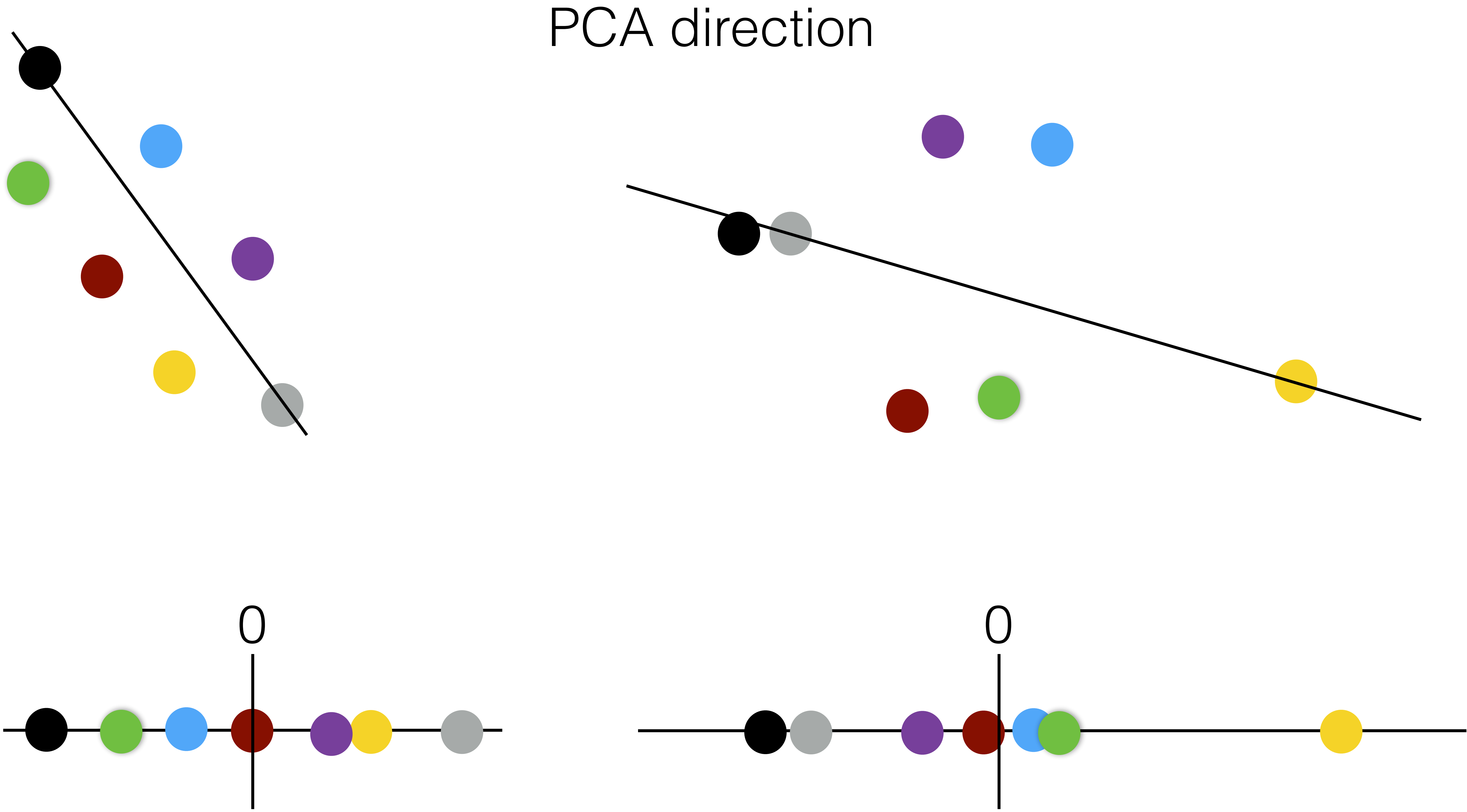

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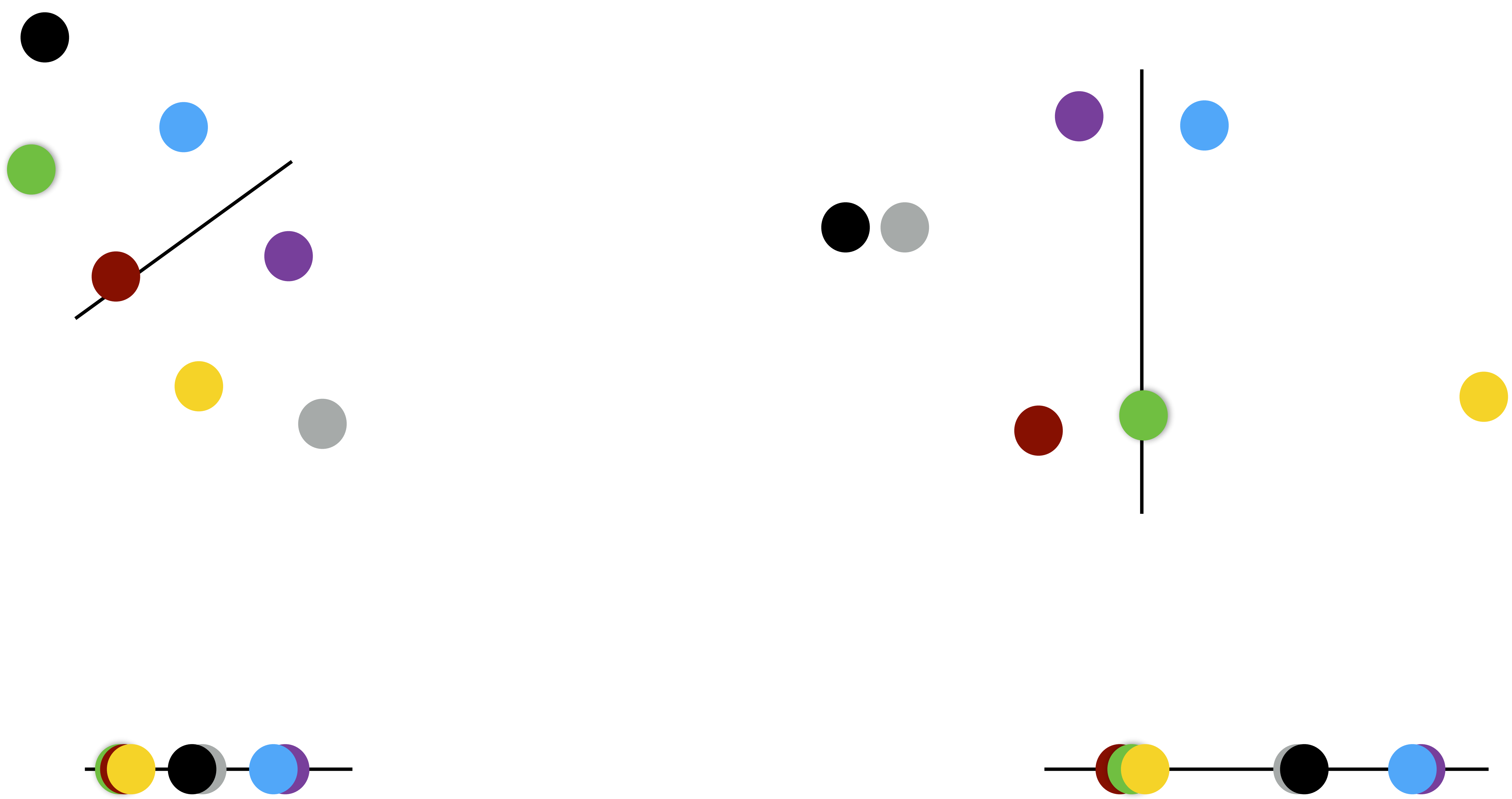

Direction has large covariance

How do we pick the right direction to project to?

Pick the directions in each view so that resultant projections have high covariance.

Maximizing Correlation Coefficient

- Say $\mathbf{w}_1$ and $\mathbf{v}_1$ are the directions we choose to project in views 1 and 2 respectively we want these directions to maximize,

$$\frac{1}{n}\sum_{t=1}^{n}\left(\mathbf{y}_t[1]-\frac{1}{n}\sum_{t=1}^{n}\mathbf{y}_t[1]\right)\cdot\left(\mathbf{y}'_t[1]-\frac{1}{n}\sum_{t=1}^{n}\mathbf{y}'_t[1]\right)$$

where $\mathbf{y}_t[1]=\mathbf{w}_1^\top\mathbf{x}_t$ and $\mathbf{y}'_t[1]=\mathbf{v}_1^\top\mathbf{x}'_t$

This should work right?!?!

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Why not Maximize Covariance

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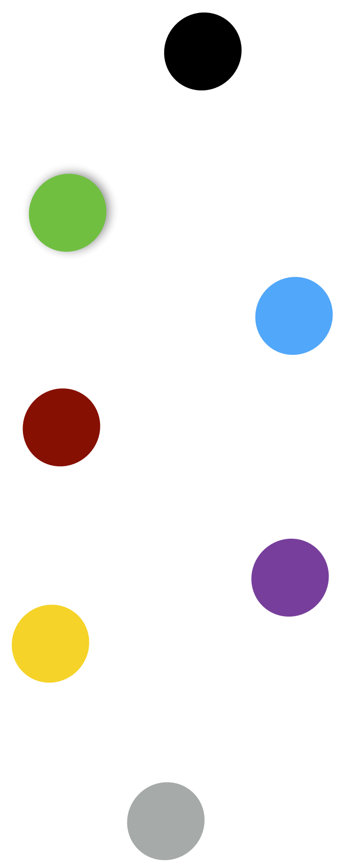

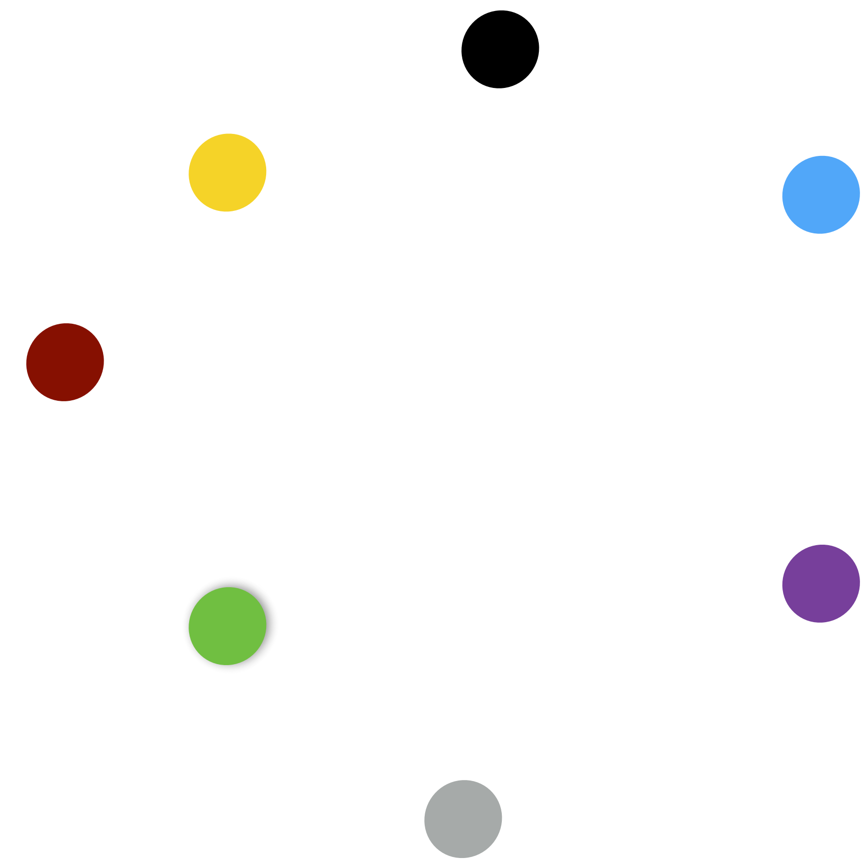

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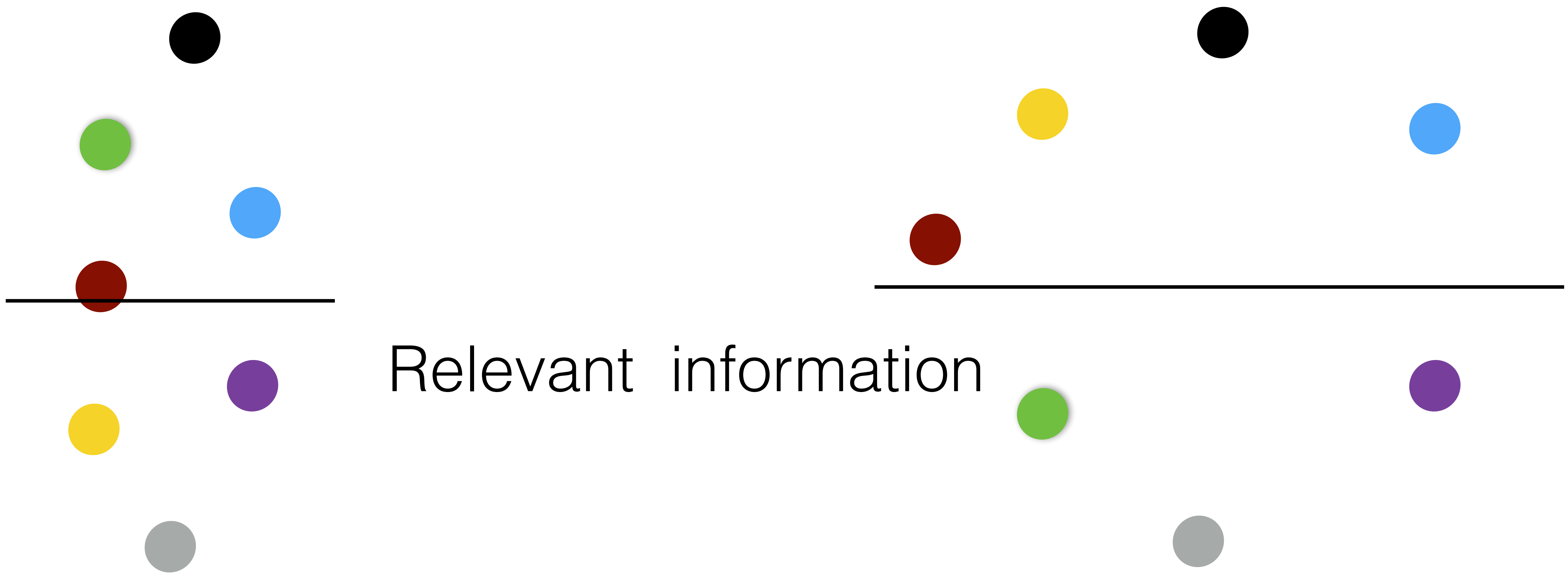

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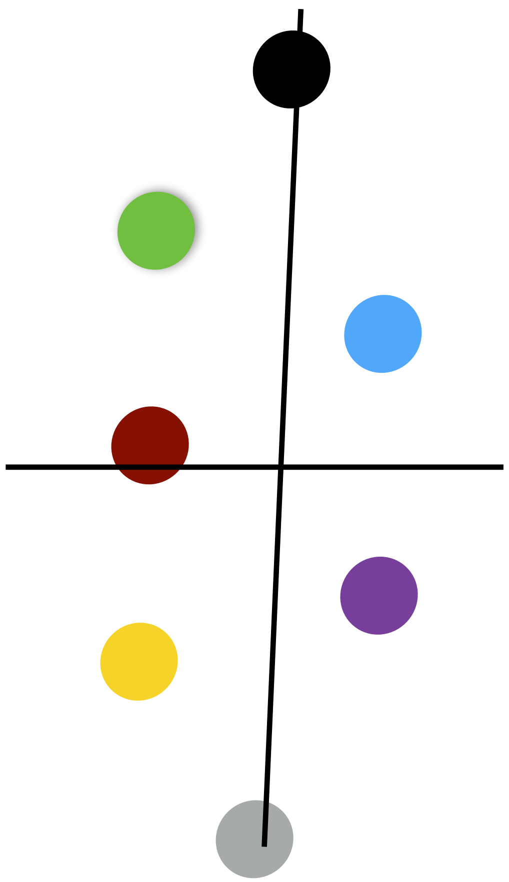
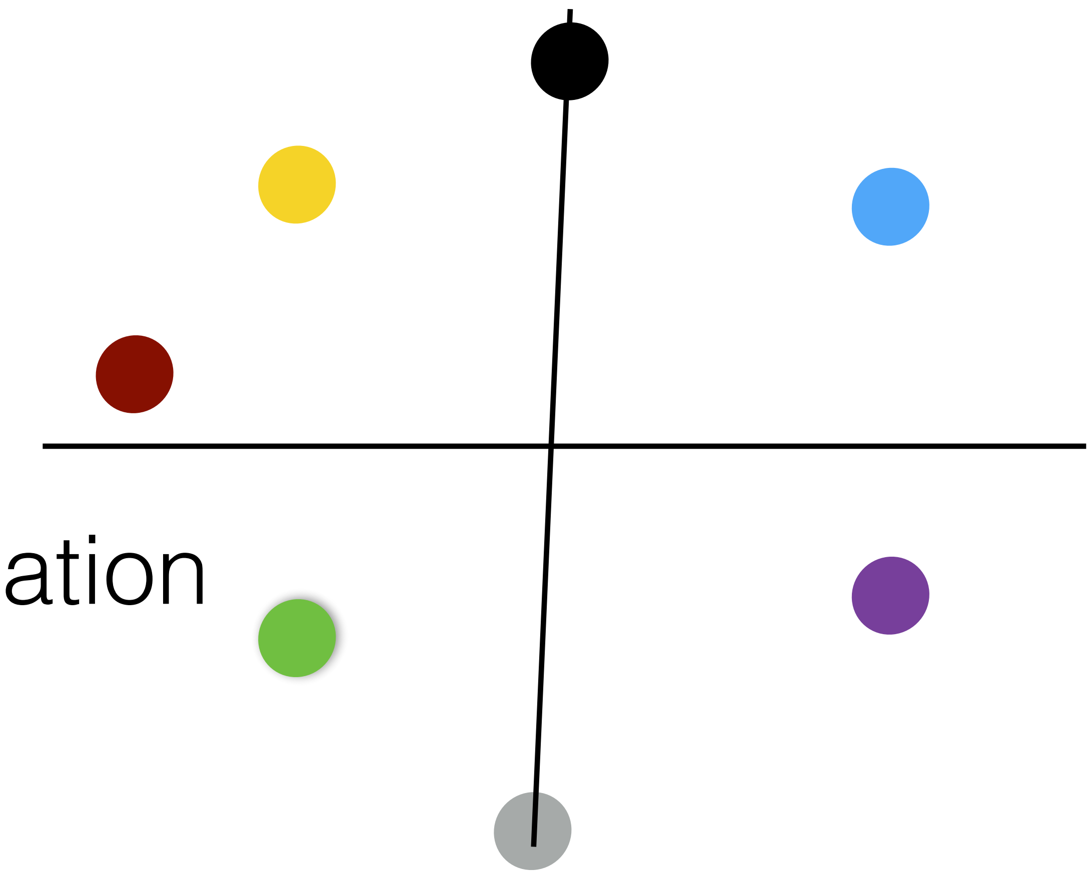


Relevant information



Say covariance in some coordinate just happens to be > 0

Scaling up this coordinate we can blow up covariance

Maximizing Correlation Coefficient

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Basic Idea of CCA

- Normalize variance in chosen direction to be constant (say 1)
- Then maximize covariance
- This is same as maximizing "correlation coefficient"

Covariance Vs Correlation

- $\text{Covariance}(A, B) = \mathbb{E}[(A - \mathbb{E}[A]) \cdot (B - \mathbb{E}[B])]$

  Depends on the scale of A and B. If B is rescaled, covariance shifts.

- $\text{Corelation}(A, B) = \frac{\mathbb{E}[(A-\mathbb{E}[A]) \cdot (B-\mathbb{E}[B])]}{\sqrt{\text{Var}(A)}\sqrt{\text{Var}(B)}}$

  Scale free.

Maximizing Correlation Coefficient

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s.t. $\frac{1}{n}\sum_{t=1}^{n}\left(\mathbf{y}_t[1] - \frac{1}{n}\sum_{t=1}^{n}\mathbf{y}_t[1]\right)^2 = \frac{1}{n}\sum_{t=1}^{n}\left(\mathbf{y}'_t[1] - \frac{1}{n}\sum_{t=1}^{n}\mathbf{y}'_t[1]\right) = 1$

where $\mathbf{y}_t[1] = \mathbf{w}_1^\top \mathbf{x}_t$ and $\mathbf{y}'_t[1] = \mathbf{v}_1^\top \mathbf{x}'_t$

Canonical Correlation Analysis

- Hence we want to solve for projection vectors $\mathbf{w}_1$ and $\mathbf{v}_1$ that

$$\text{maximize } \frac{1}{n}\sum_{t=1}^{n} \mathbf{w}_1^\top(\mathbf{x}_t - \mu) \cdot \mathbf{v}_1^\top(\mathbf{x}_t' - \mu')$$

$$\text{subject to } \frac{1}{n}\sum_{t=1}^{n} (\mathbf{w}_1^\top(\mathbf{x}_t - \mu))^2 = \frac{1}{n}\sum_{t=1}^{n} (\mathbf{v}_1^\top(\mathbf{x}_t' - \mu'))^2 = 1$$

where $\mu = \frac{1}{n}\sum_{t=1}^{n} \mathbf{x}_t$ and $\mu' = \frac{1}{n}\sum_{t=1}^{n} \mathbf{x}_t'$

Canonical Correlation Analysis

- Hence we want to solve for projection vectors $\mathbf{w}_1$ and $\mathbf{v}_1$ that

$$\begin{aligned}\text{maximize } & \mathbf{w}_1^\top \Sigma_{1,2} \mathbf{v}_1 \\ \text{subject to } & \mathbf{w}_1^\top \Sigma_{1,1} \mathbf{w}_1 = \mathbf{v}_1^\top \Sigma_{2,2} \mathbf{v}_1 = 1\end{aligned}$$

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$$\Sigma = \begin{matrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{matrix} = \mathrm{cov}\left(X \; X' \right)$$

Solution

V

Solution

$$W = \operatorname{eigs}\left(\Sigma_{11}^{-1}\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}, K\right)$$

$$V = \operatorname{eigs}\left(\Sigma_{22}^{-1}\Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12}, K\right)$$

CCA Algorithm

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3. $\boxed{W} = \mathrm{eigs}\left(\boxed{\Sigma_{11}^{-1}\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}}, K \right)$

4. $\boxed{Y_1} = \boxed{X_1 - \mu_1} \times \boxed{W}$

CCA Demo