Machine Learning for Data Science (CS4786) Lecture 6

Random Projections

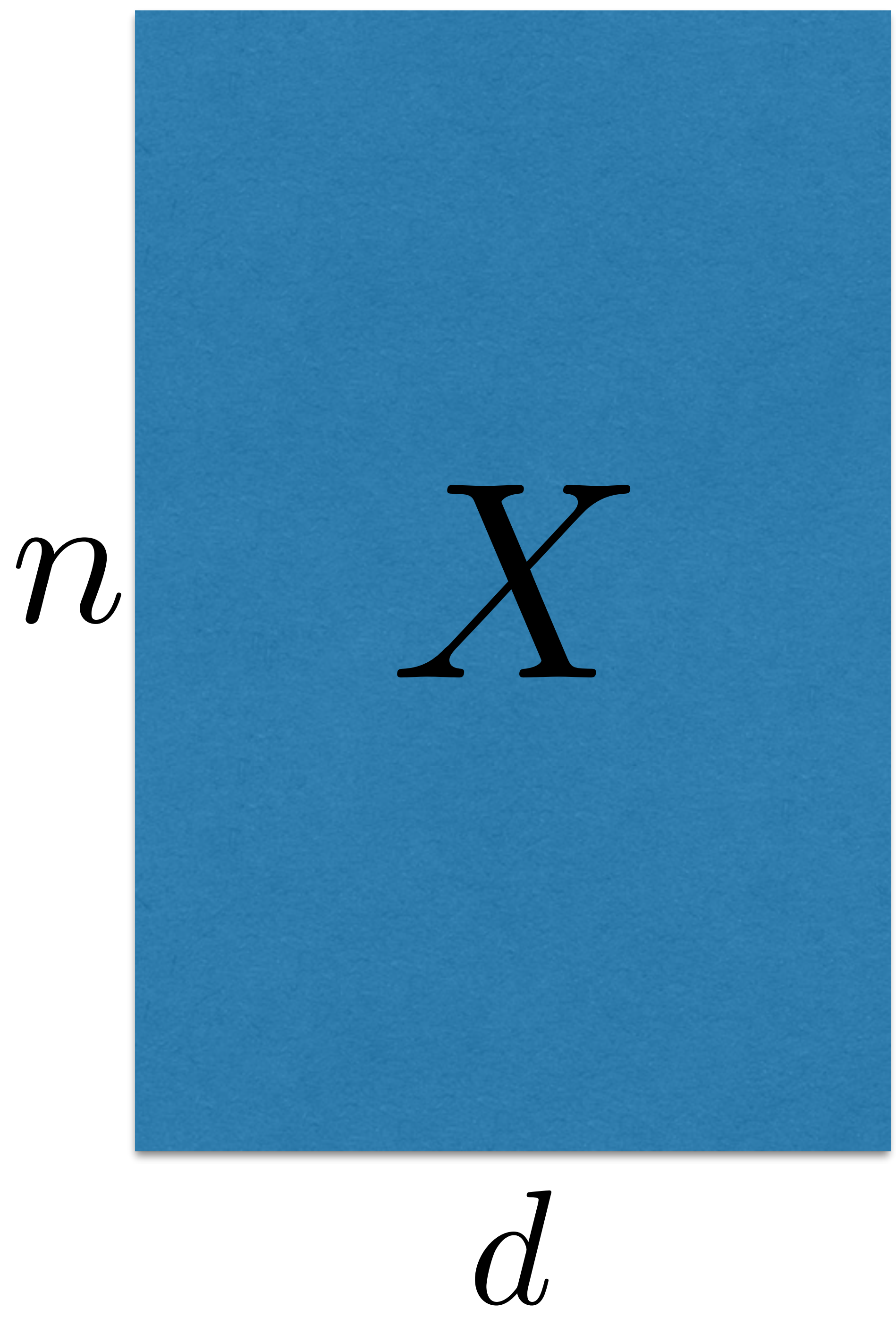
Feb 09, 2015

Course Webpage :
http://www.cs.cornell.edu/Courses/cs4786/2015sp/

CCA Demo Redo?

BACK TO SINGLE VIEW: RECAP

Back to Single View: Recap



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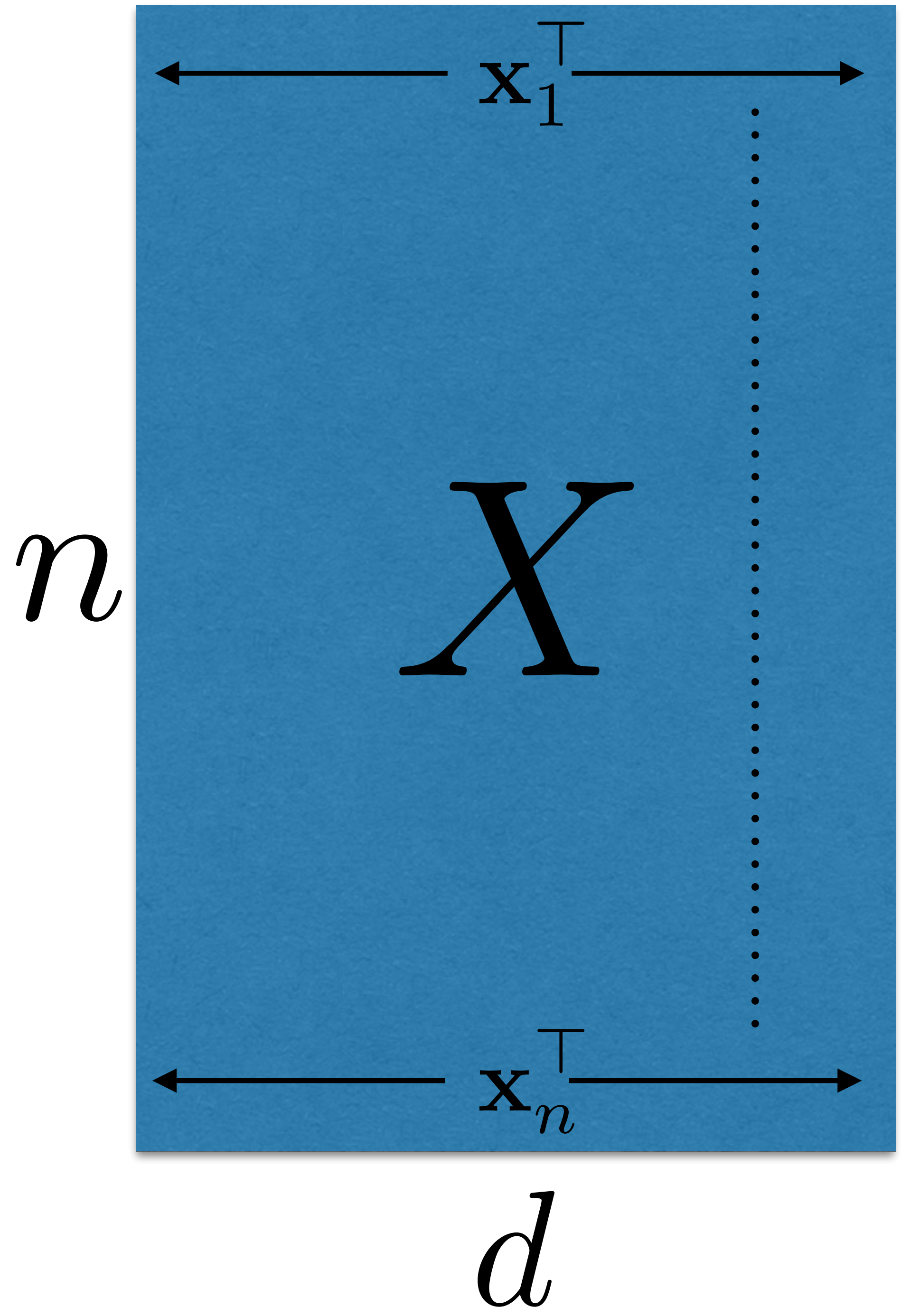

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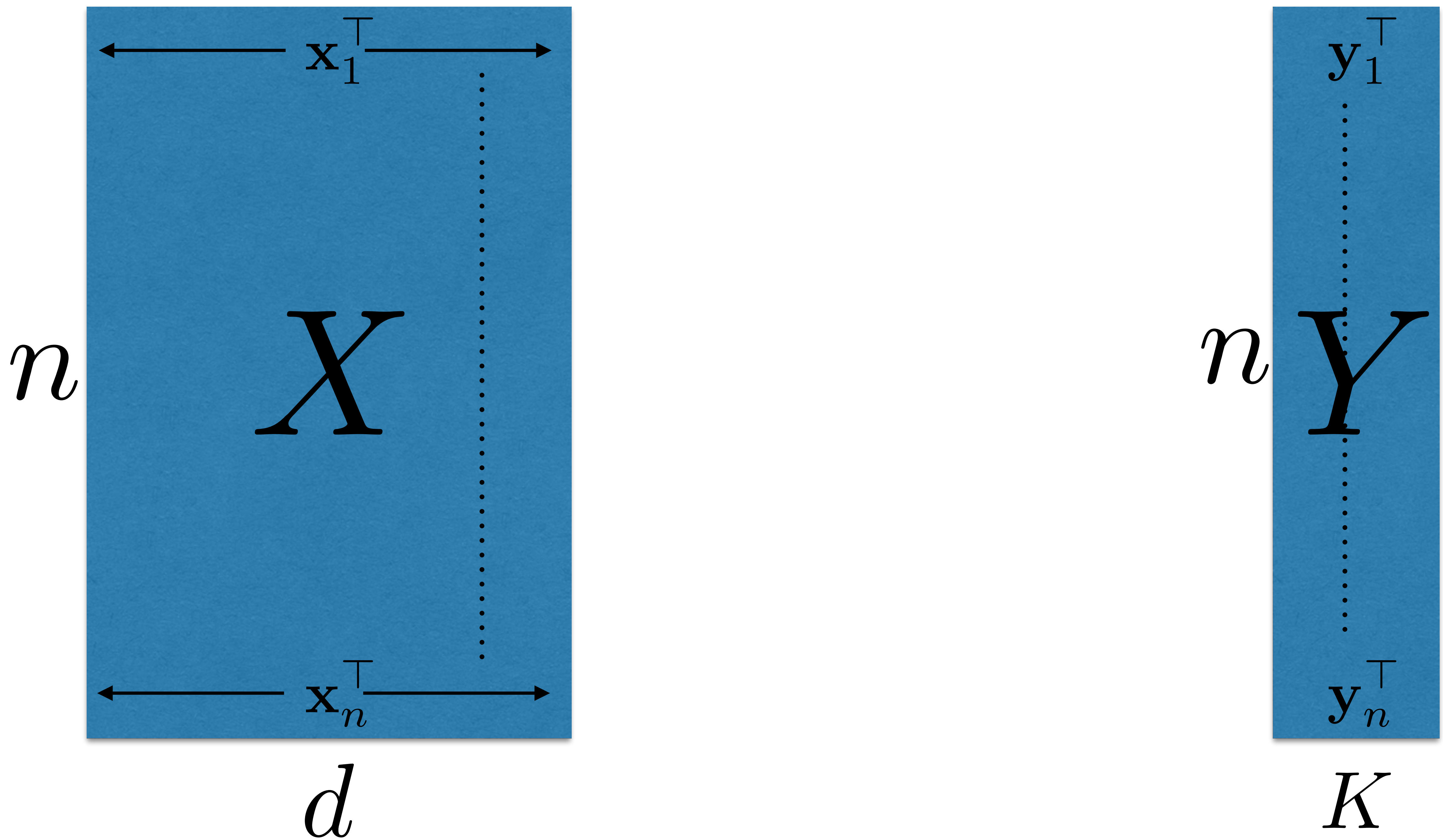

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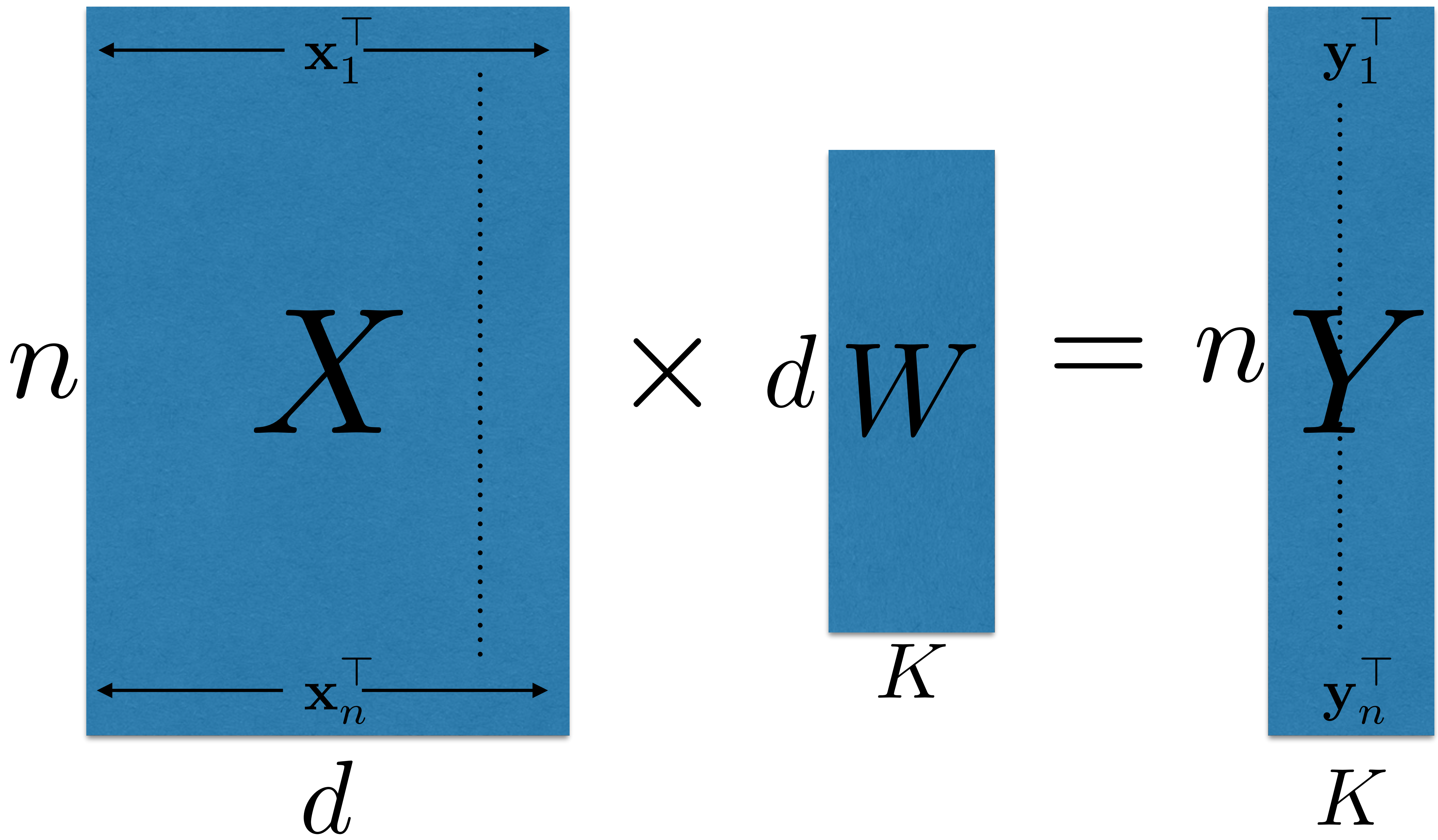
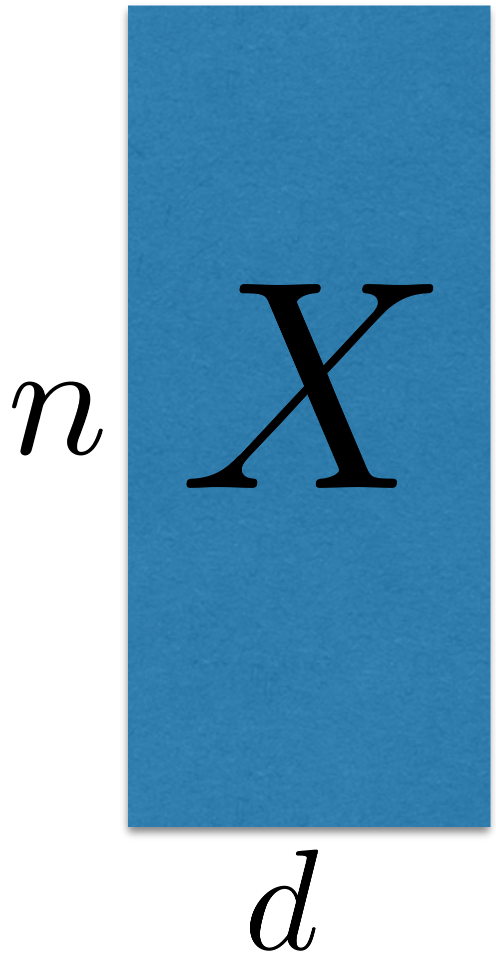


The Tall, the Fat and the Ugly



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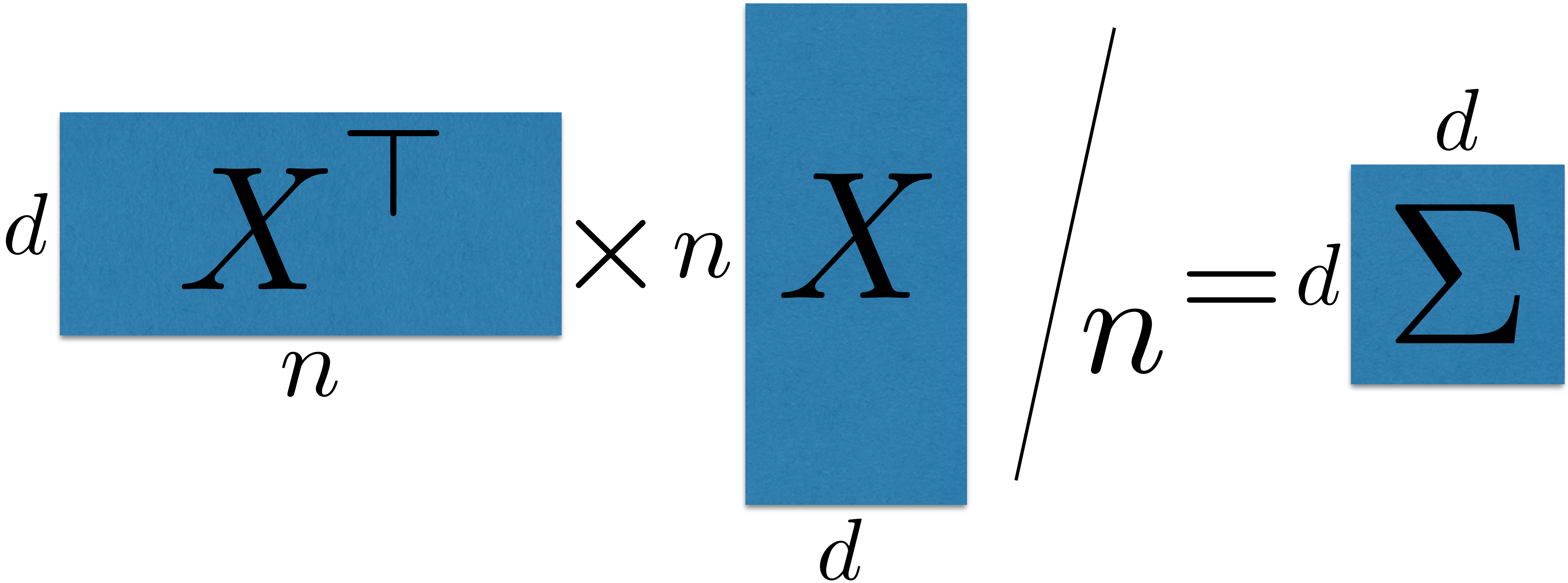
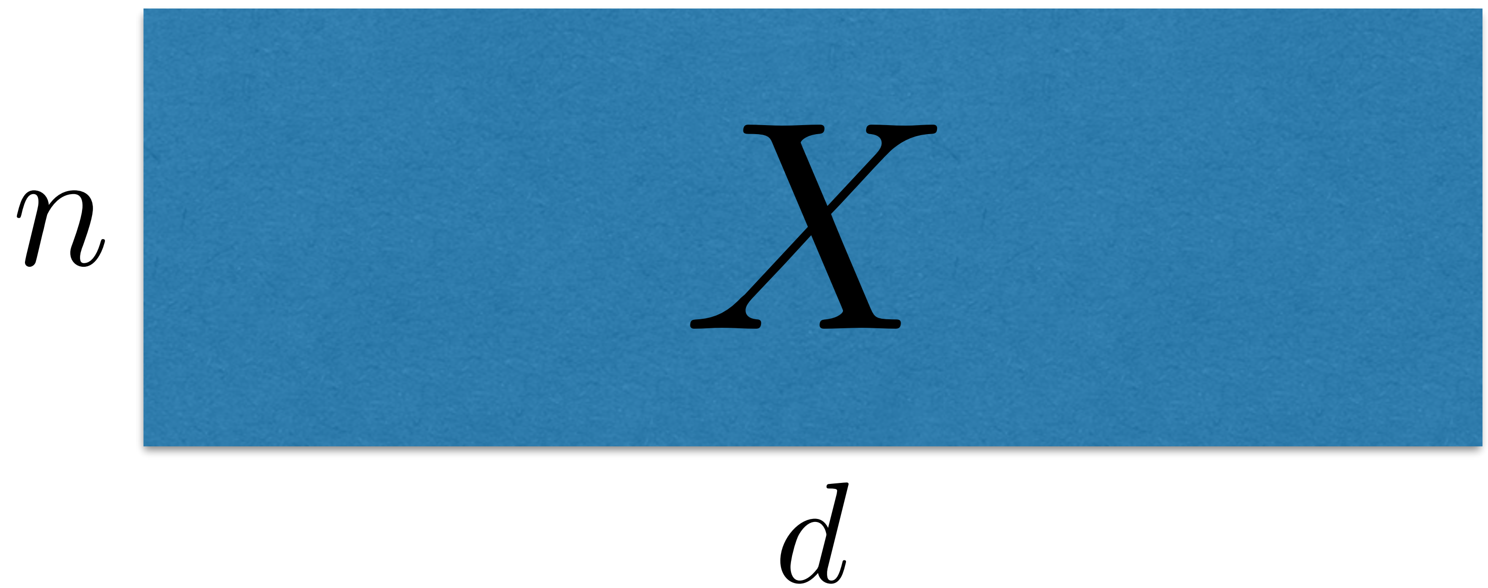


The Tall, the Fat and the Ugly

$$\underset{d \times n}{X^\top} \times \underset{n \times d}{X} \Big/ n = \underset{d \times d}{\Sigma}$$

$$\underset{d \times K}{W} = \mathrm{Eigs}(\Sigma, K)$$

The Tall, **the Fat** and the Ugly



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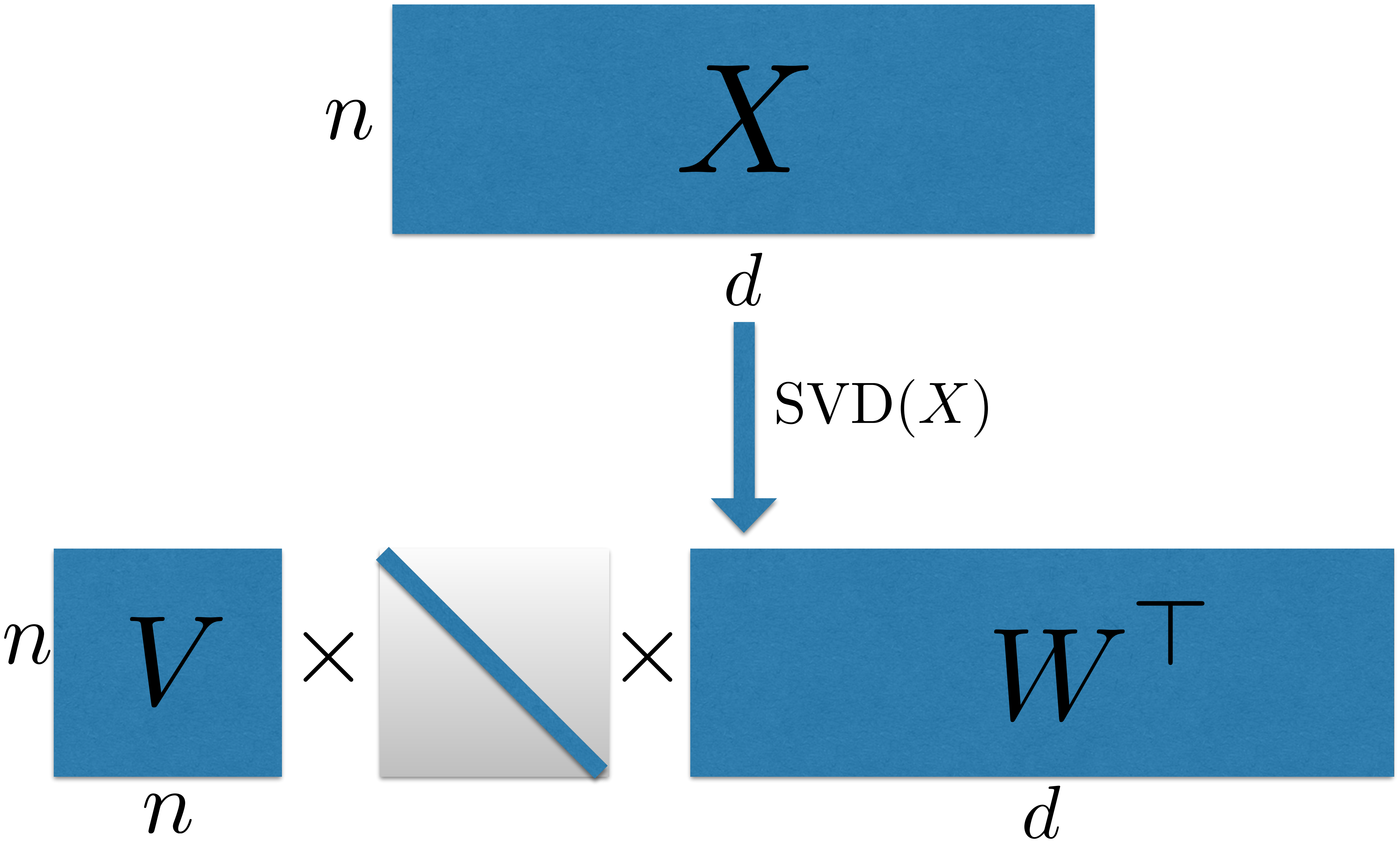
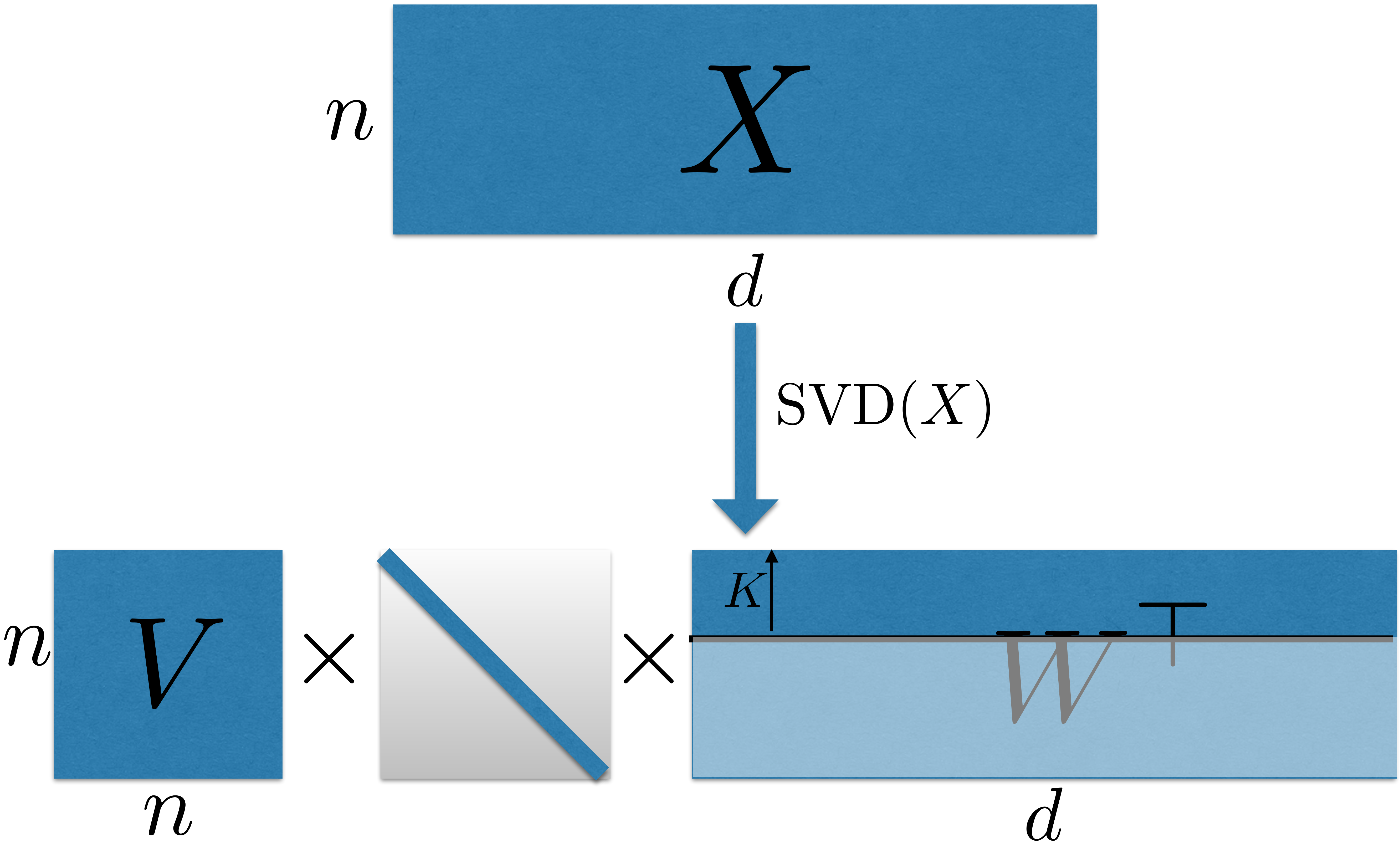
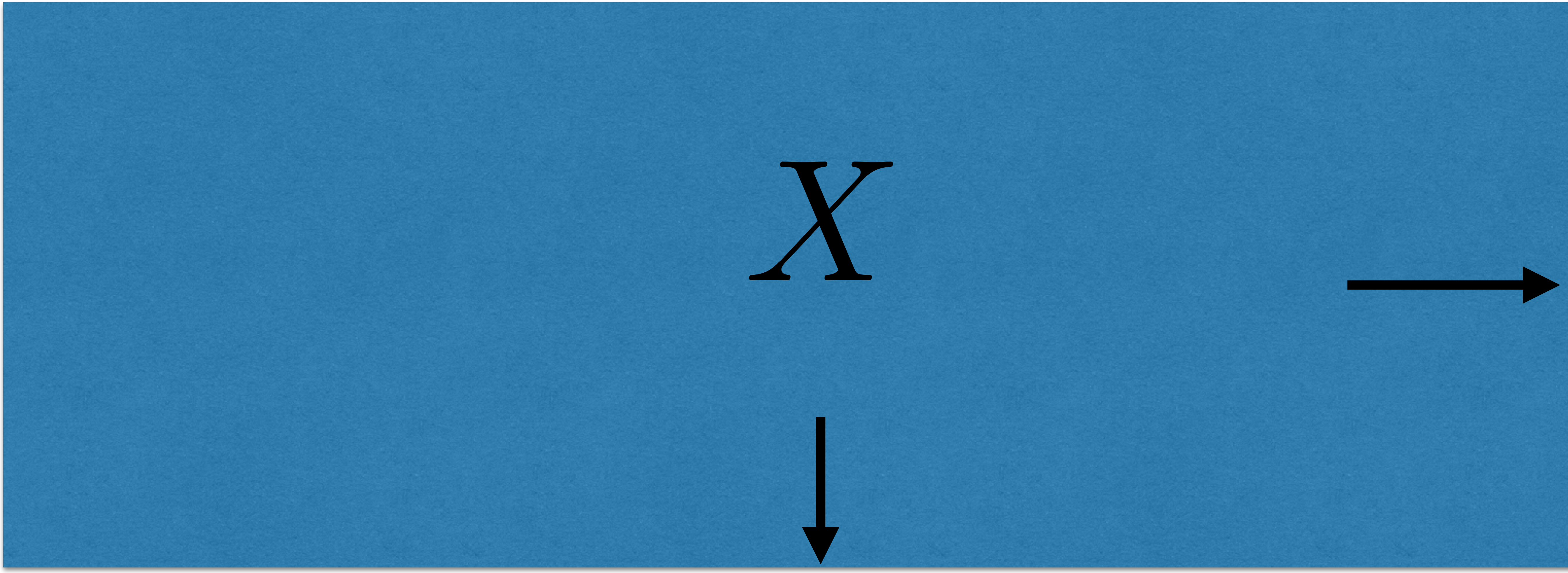




n
X
d
SVD(X)
n
V
n
×
×
K
W⊤
d

The Tall, the Fat and **the Ugly**



- d and n so large we can't even store in memory
- Only have time to be linear in $\text{size}(X) = n \times d$

I there any hope?

Pick a Random W

Pick a Random W

$$Y = X \times \left.\left[\begin{array}{ccc} +1 & \dots & -1 \\ -1 & \dots & +1 \\ +1 & \dots & -1 \\ & \cdot & \\ & \cdot & \\ & \cdot & \\ +1 & \dots & -1 \end{array}\right]_{K}^{d} \middle/ \sqrt{K} \right.$$

Why should Random Projections even work?!

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  Distances between all pairs of data-points in low dim. projection is roughly the same as their distances in the high dim. space.

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Distances between all pairs of data-points in low dim. projection is roughly the same as their distances in the high dim. space.

That is, when K is "large enough", with "high probability", for all pairs of data points $i, j \in \{1, \ldots, n\}$,

$$(1-\epsilon)\left\|\mathbf{y}_i - \mathbf{y}_j\right\|_2 \leq \left\|\mathbf{x}_i - \mathbf{x}_j\right\|_2 \leq (1+\epsilon)\left\|\mathbf{y}_i - \mathbf{y}_j\right\|_2$$

Why should Random Projections even work?!

Consider any vector $\tilde{\mathbf{x}} \in \mathbb{R}^d$ and let $\tilde{\mathbf{y}} = W^\top \tilde{\mathbf{x}}$. Note that

$$\tilde{\mathbf{y}}[j]^2 = \left(\sum_{i=1}^{d} W[i,j] \cdot \tilde{\mathbf{x}}[i] \right)^2$$

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if $i \neq i'$, $W[i,j]$ and $W[i',j]$ are independent and so

$$= \sum_{i=1}^{d} \mathbb{E}\left[\left(W[i,j]^2\right)\right] \tilde{\mathbf{x}}[i]^2 + \sum_{i \neq i'} \left(\mathbb{E}[W[i,j]] \cdot \mathbb{E}\left[W[i',j]\right]\right) \cdot \left(\tilde{\mathbf{x}}[i] \cdot \tilde{\mathbf{x}}[i']\right)$$

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$$= \sum_{i=1}^{d} \tilde{\mathbf{x}}[i]^2/\sqrt{K}^2 = \|\tilde{\mathbf{x}}\|_2^2/K$$

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Hence for any $s, t \in \{1, \ldots, n\}$,

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Lets try this in Matlab …

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For any $\epsilon > 0$, if $K \approx \log(n/\delta)/\epsilon^2$, with probability $1 - \delta$ over draw of W, for all pairs of data points $i, j \in \{1, \ldots, n\}$,

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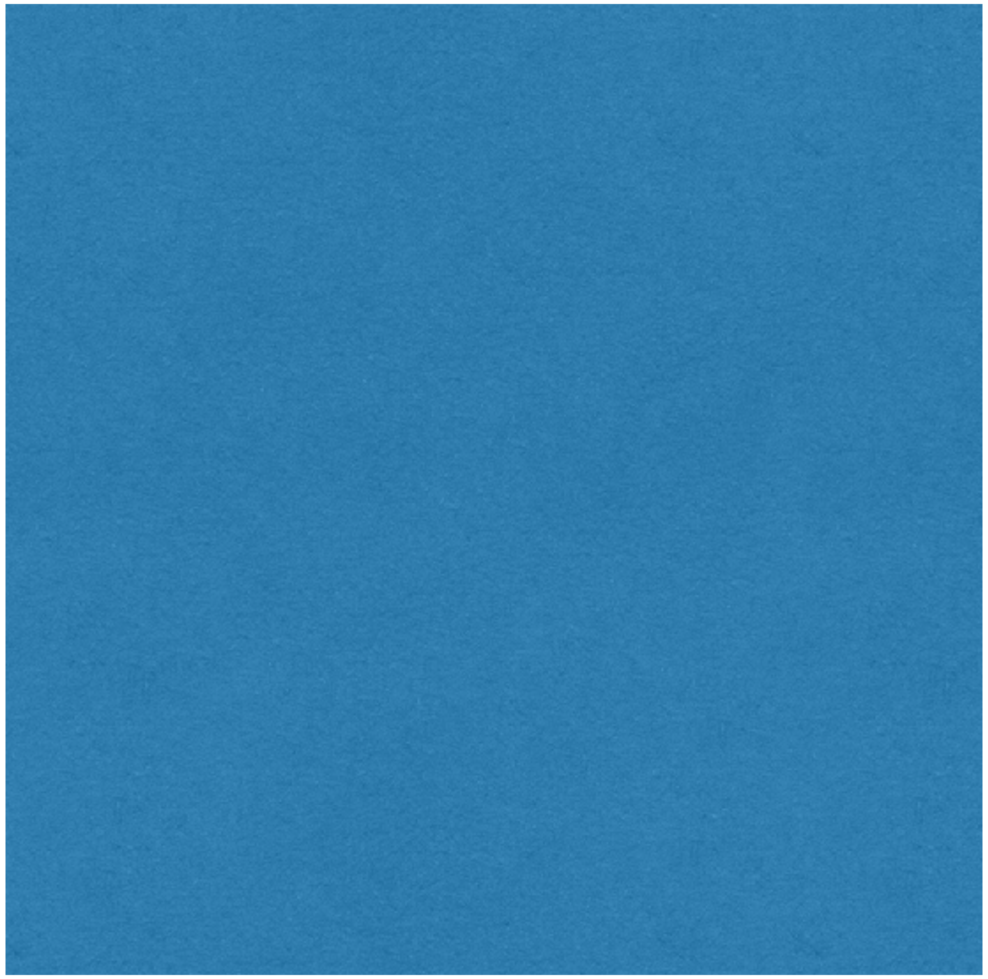
This is called the Johnson-Lindenstrauss lemma or JL lemma for short.

Why is this so Ridiculously Magical?

n=
1000



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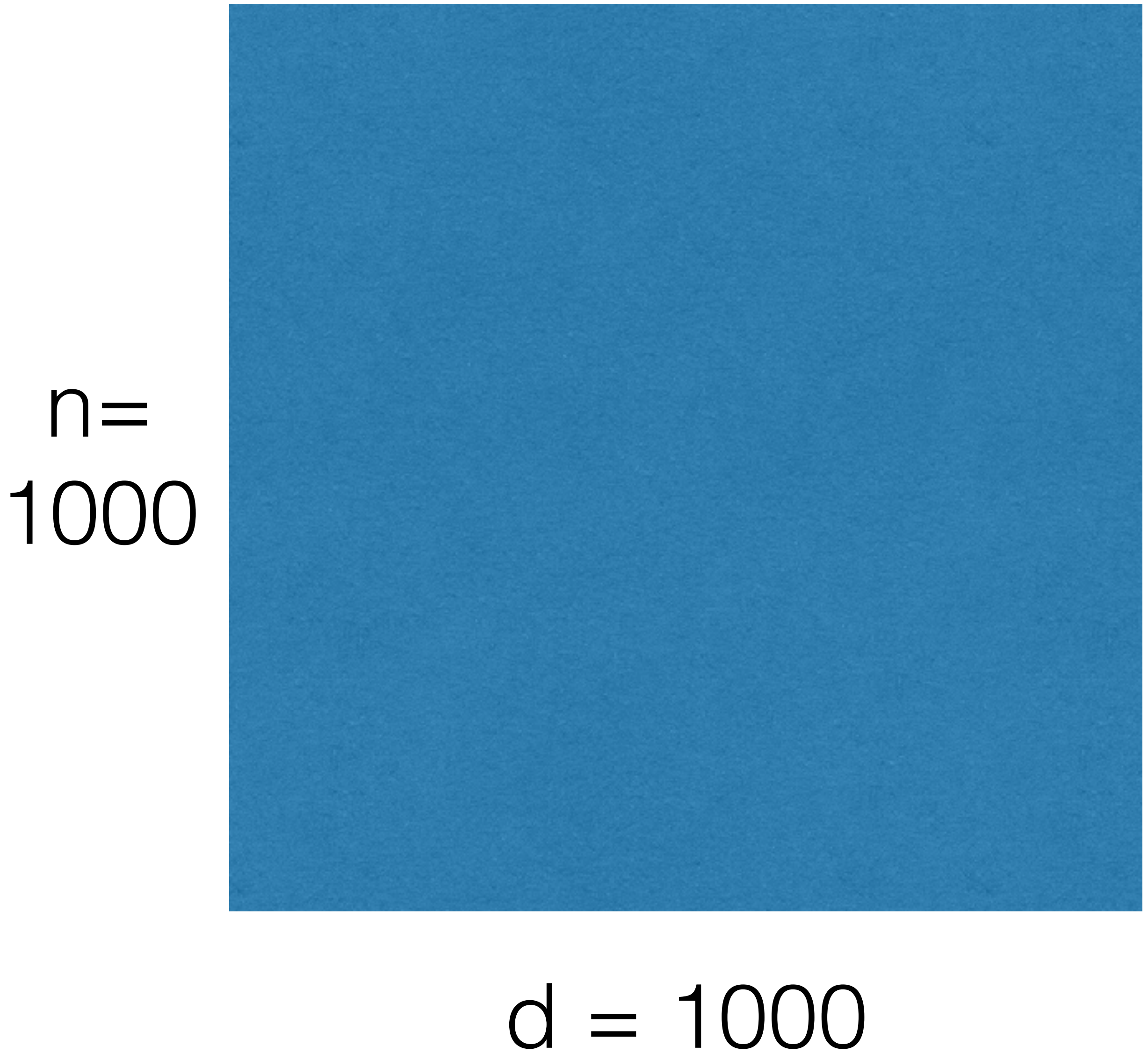

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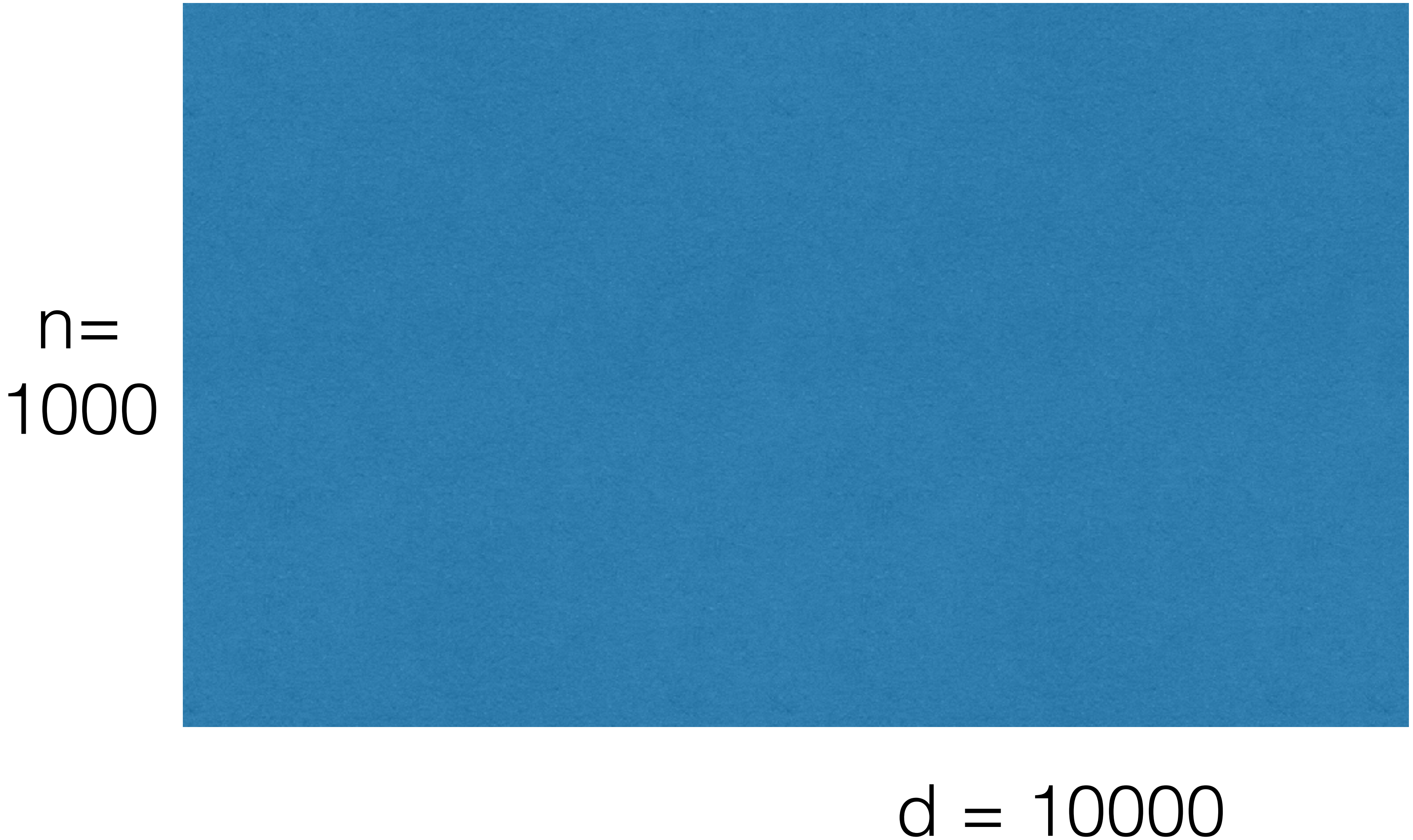

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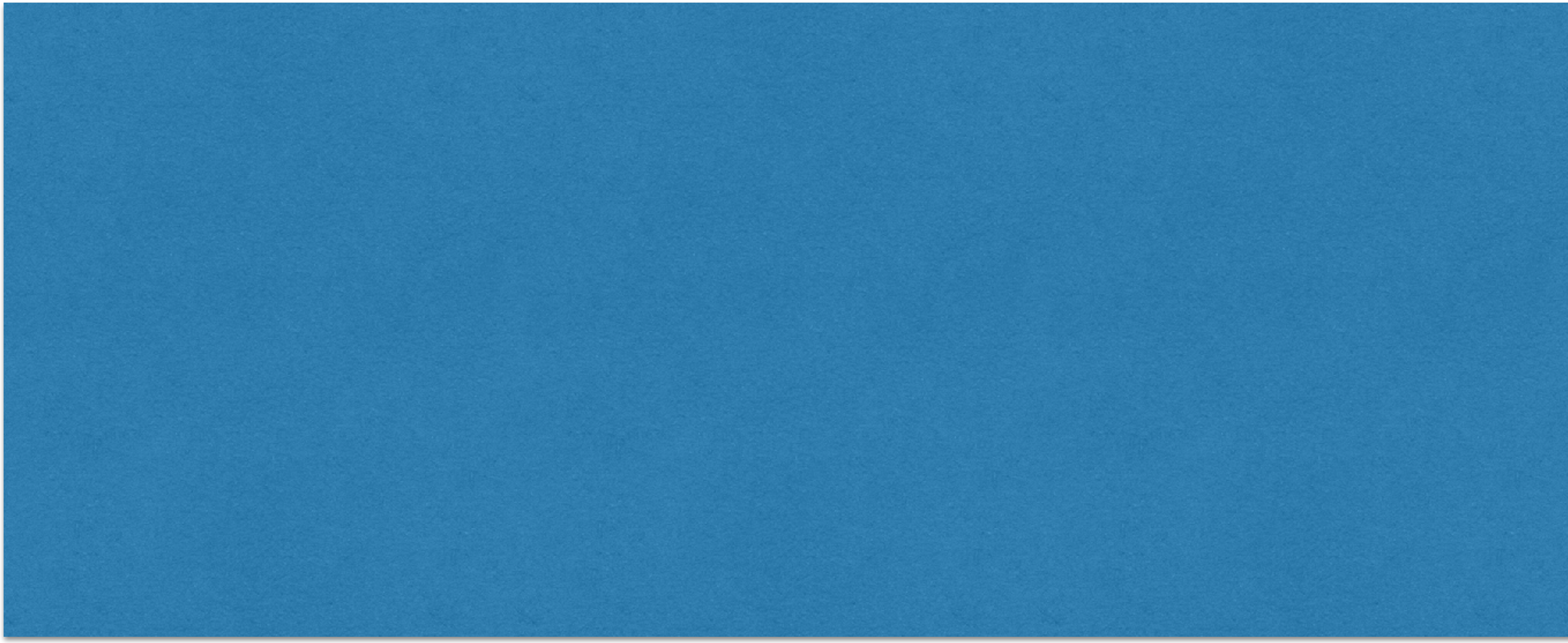

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