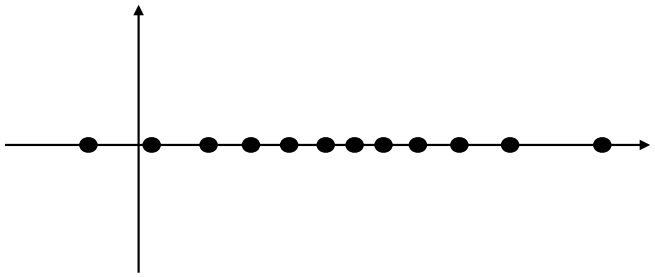
Bayes Classifier and Naive Bayes

Announcements

HW 2 is out — start early

Recap on MLE



$$\mathscr{D} = \{x_i\}_{i=1}^{n}, x_i \in \mathbb{R}$$

Assume data is from $\mathcal{N}(\mu^\star, \sigma^2)$, want to estimate $\mu^\star, \sigma$ from the data $\mathscr{D}$ MLE

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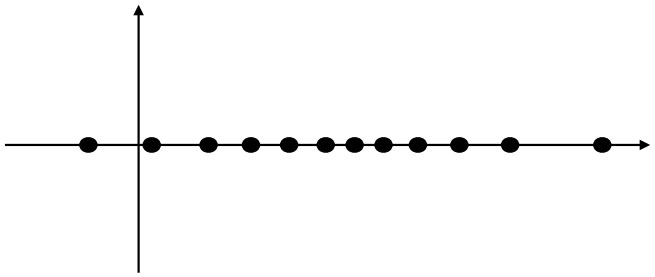

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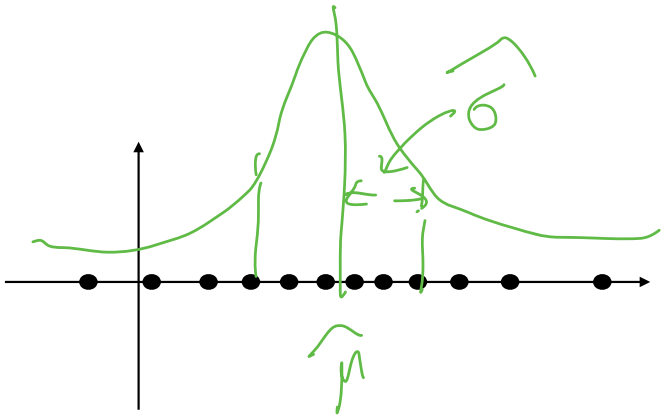
$P(x \mid \mu, \sigma)$

MLE:

$\underset{\mu,\sigma}{\operatorname{argmax}}\ P(\mathcal{D} \mid \mu, \sigma)$

Recap on MLE



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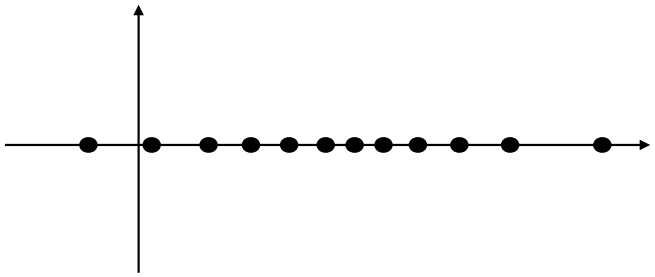
The solution that maximizes the log-likelihood:

$$\hat{\mu} = \sum_{i=1}^{n} x_i/n, \quad \hat{\sigma}^2 = \sum_{i=1}^{n} (x_i - \hat{\mu})^2/n$$

Recap on MAP



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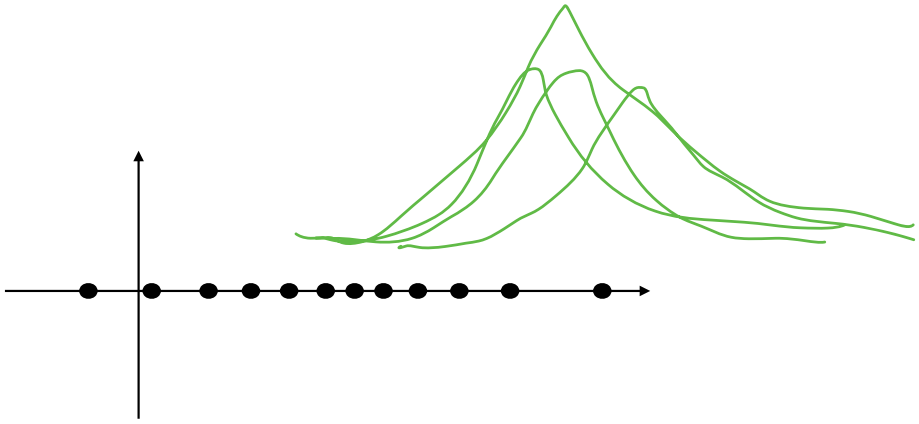

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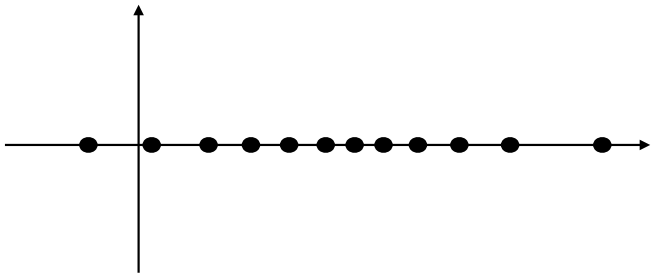

$$\mathscr{D} = \{x_i\}_{i=1}^{n}, x_i \in \mathbb{R}$$

Now if we want to use **MAP:**

1. Pick a prior: $P(\mu, \sigma)$

2. Write down data-likelihood: $P(\mathscr{D} \mid \mu, \sigma) = \prod_{i=1}^{n} \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2}(x_i - \mu)^2/\sigma^2\right)$

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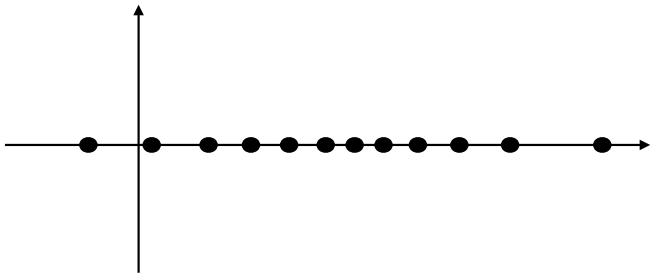

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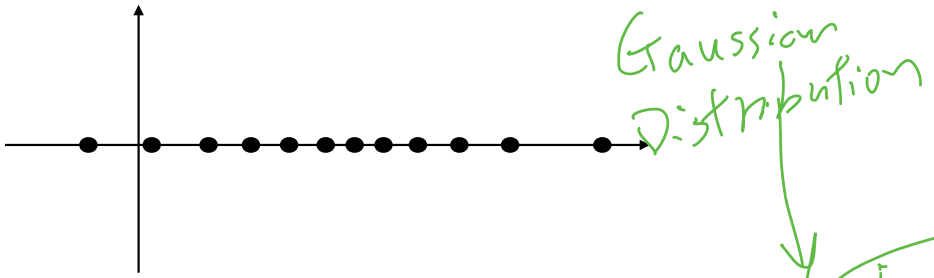

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Gaussian Distribution

Now if we want to use **MAP:**

$P(\mu)P(\sigma)$ ← Inverse Gamma

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Today

Objective: learn our second classification algorithm—Naive Bayes (derived via MLE)

Outline

1. General formulation of Naive Bayes

2. Example

3. Connection to linear classifier

Generative modeling

Setting: binary classification w/ dataset $\{x_i, y_i\}_{i=1}^n$, $(x_i, y_i) \sim P$, where $x \in \mathbb{R}^d$, $y \in \{-1, 1\}$

Goal: estimate $P(y \mid x)$

$x_i \in \mathbb{R}^d$

$\hat{P}(y \mid x)$ from D

Ex. $\hat{P}(y=1 \mid x=x_0)$

$\operatorname{argmax}_{y \in \{-1,+1\}} P(y \mid x)$

$$P(y \mid x) = \frac{P(x,y)}{P(x)} = \frac{\sum_{i=1}^n 1(x_i = x, y_i = y)}{\sum_{i=1}^n 1(x_i = x)}$$

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Bayes Rule

$$P(y \mid x) \propto P(x \mid y) P(y)$$

$$P(y \mid x) = \frac{P(x \mid y) P(y)}{P(x)}$$

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(Discriminative modeling: directly estimate $P(y \mid x)$)

Naive Bayes

Estimate $P(y)$ from data:

$$P(y \mid x) \propto P(x \mid y) P(y)$$

$y = \{+1, -1\}$

Naive Bayes

Estimate $P(y)$ from data:

$$P(y \mid x) \propto P(x \mid y)P(y)$$

Estimate $P(y)$ is easy:

$$P(y = 1) \approx \frac{\sum_{i=1}^{n} \mathbf{1}(y_i = 1)}{n}$$

$P(y=-1) = \frac{\sum_{i=1}^{n} 1(y_i = -1)}{n}$

Naive Bayes

Estimate $P(x \mid y)$ from data:

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Naive Bayes

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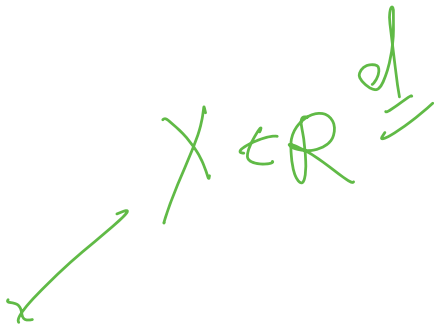

$$P(y \mid x) \propto P(x \mid y)P(y)$$

Estimate $P(x \mid y)$ is not easy:

x can be high-dimensional, e.g., d is large!

There may not be repetitions in $\{x_i\}_{i=1}^{n}$!

The key assumption in Naive Bayes

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About the independence assumption

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Q: does conditional independence imply global independence?

$$P(x) = \prod_{i=1}^{d} P(x[i])$$

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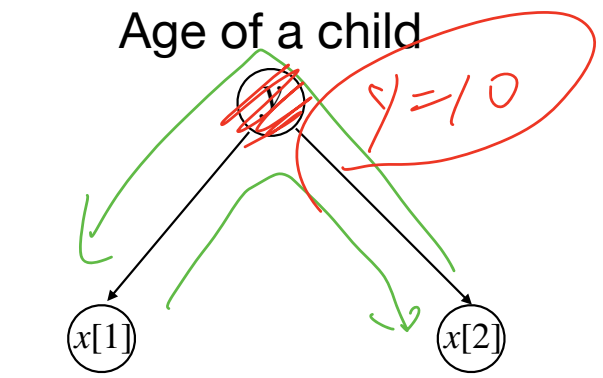

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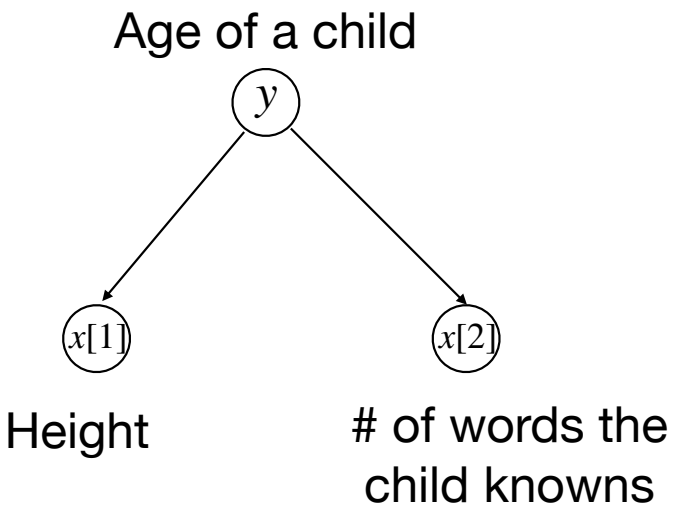

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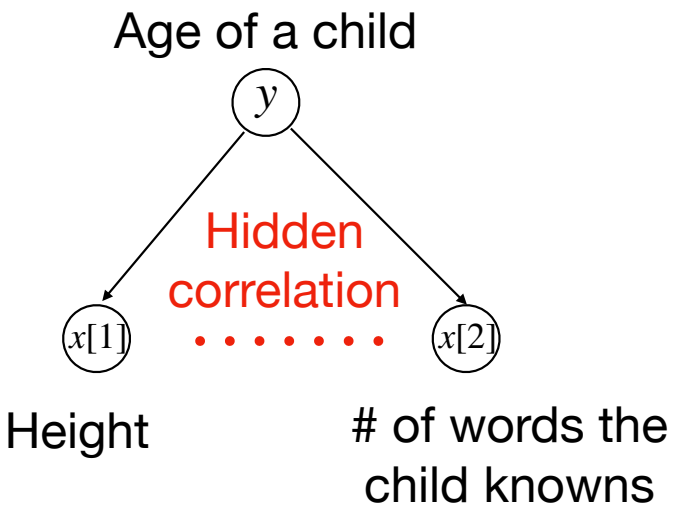

Naive Bayes

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Naive Bayes

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1-D scalar

W/ the NB assumption $P(x \mid y) = \prod_{\alpha=1}^{d} P(x[\alpha] \mid y)$

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Now we can estimate $P(x[\alpha] \mid y)$ for each α

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1-dim problem!

Naive Bayes

Once estimated $P(y)$ and $P(x \mid y)$, we can make prediction:

$$P(y \mid x) \propto P(x \mid y)P(y)$$

In test time, given x:

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$$= \arg\max_y P(x \mid y)P(y) = \arg\max_y \left(\prod_{\alpha=1}^{d} P(x[\alpha] \mid y)\right)P(y)$$

NB Assumption

$$= \arg\max_y \sum_{\alpha=1}^{d} \ln P(x[\alpha] \mid y) + \ln P(y)$$

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Case study

$X \in \mathbb{R}^d$

Continuous features

$x[\alpha] \in \mathbb{R}$, for all $\alpha \in \{1,2,\ldots d\}$

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$x[\alpha] \in \mathbb{R}$, for all $\alpha \in \{1,2,\ldots d\}$

We model each $P(x[\alpha] \mid y)$ using a 1-dim Gaussian distribution:

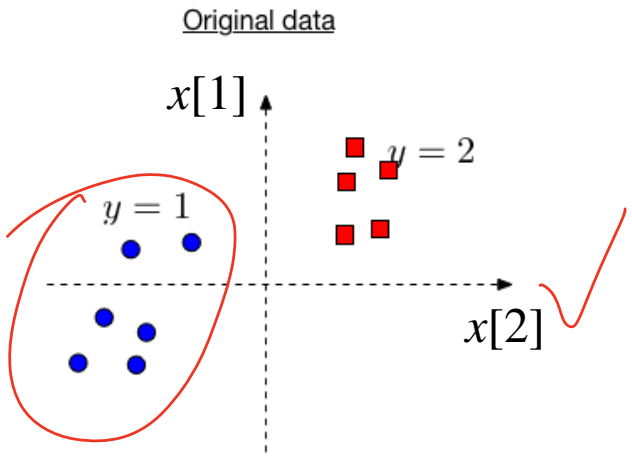
$$P(x[\alpha] \mid y) = \mathcal{N}(\mu_{\alpha,y}, \sigma^2_{\alpha,y})$$

Case study

Estimate the mean/std parameter $\mu_{\alpha,y}, \sigma_{\alpha,y}$:

$$P(x[\alpha] \mid y) = \mathcal{N}(\mu_{\alpha,y}, \sigma^2_{\alpha,y})$$

P(x|y)



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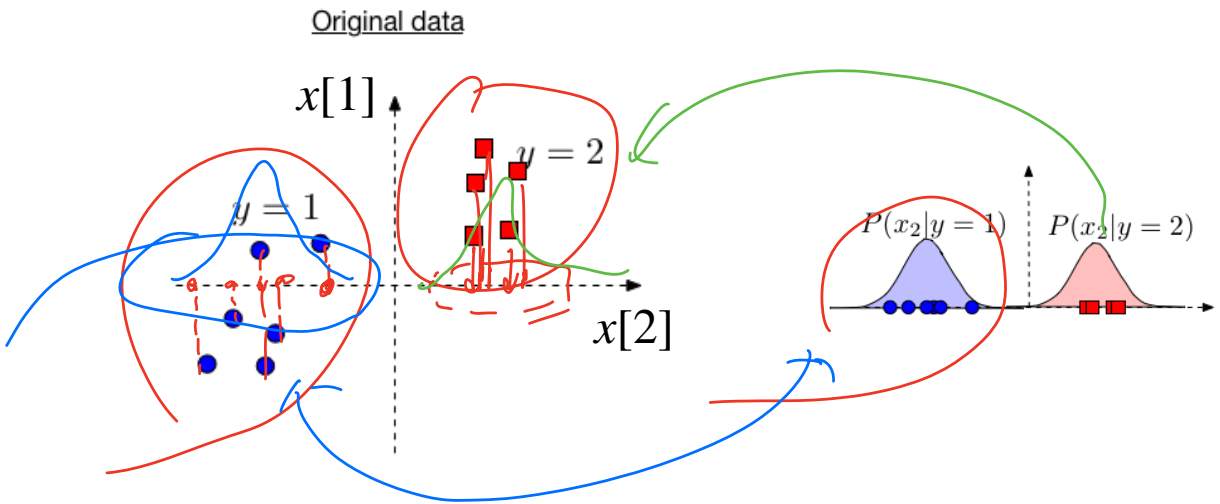

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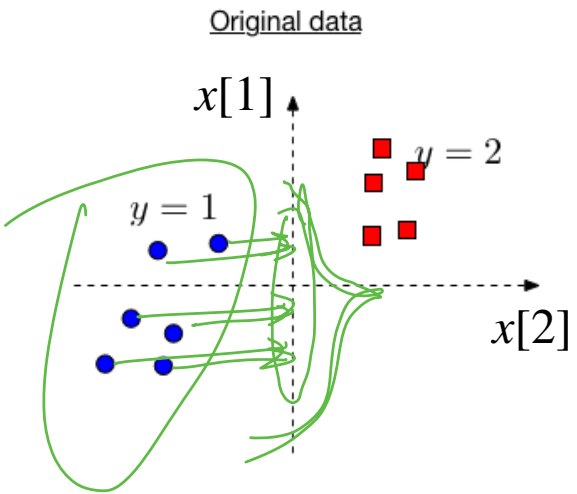

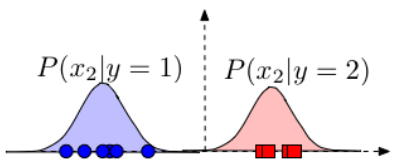

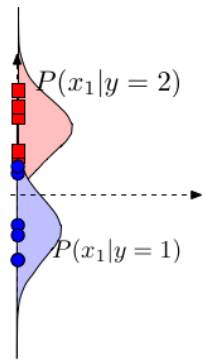

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Estimate the mean/std parameter $\mu_{\alpha,y}, \sigma_{\alpha,y}$ via **MLE**:

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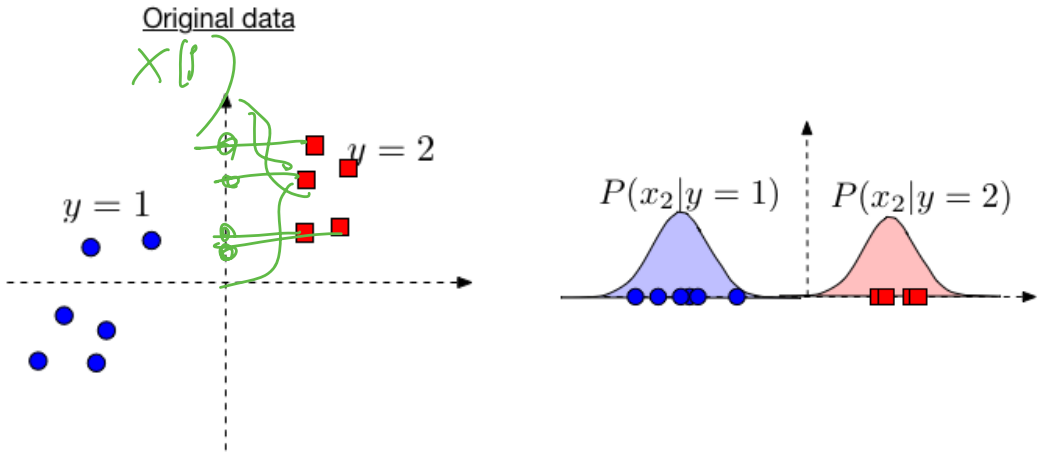

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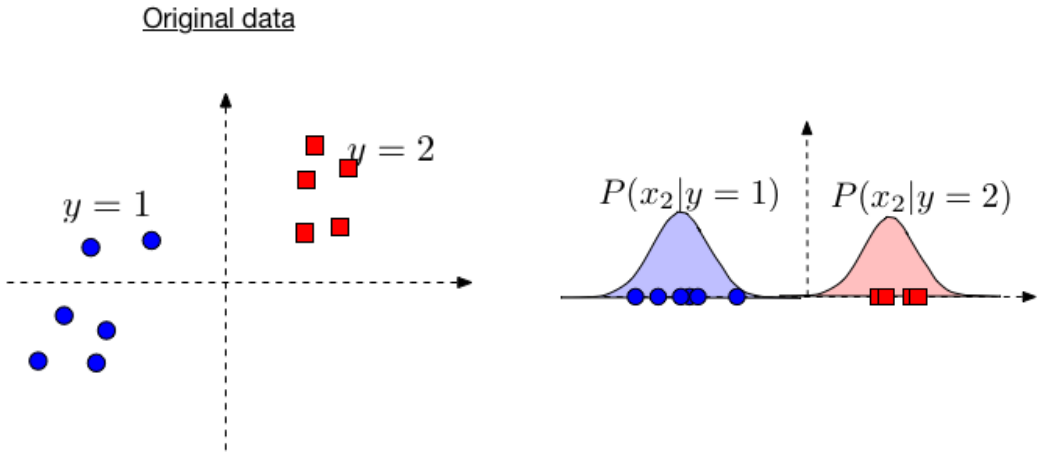

$$\mu_{\alpha,y} = \frac{\sum_{i=1}^{n} x_i[\alpha]\mathbf{1}(y_i = y)}{\sum_{i=1}^{n} \mathbf{1}(y_i = y)}$$

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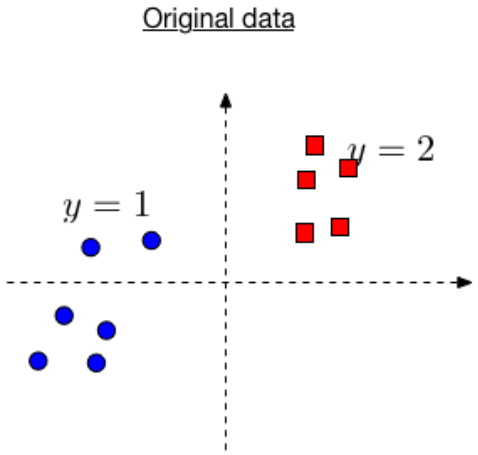
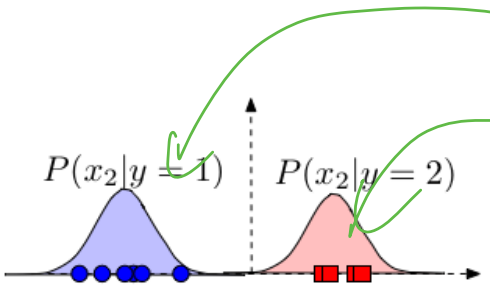




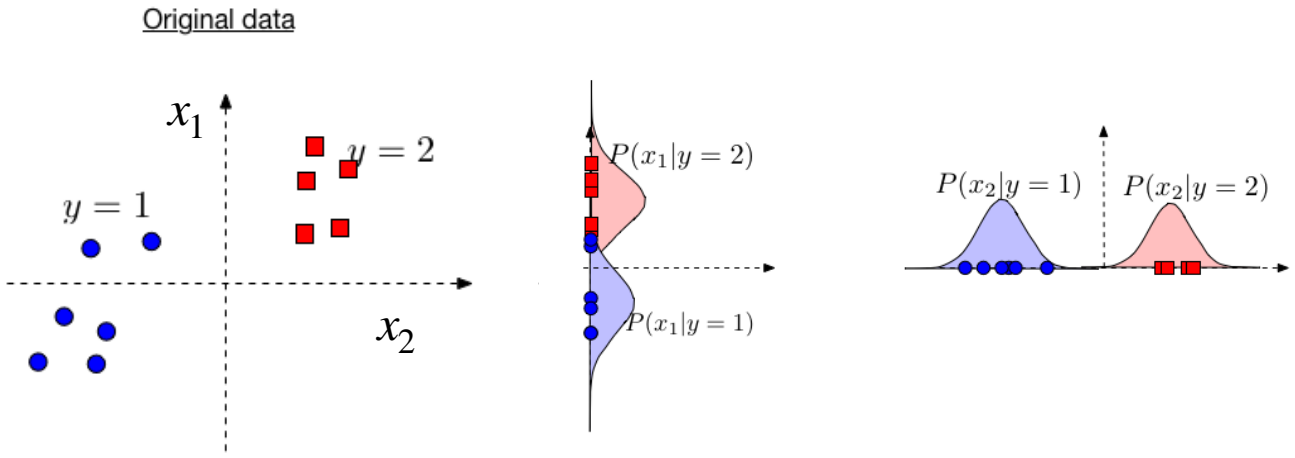
$$\mu_{\alpha,y} = \frac{\sum_{i=1}^{n} x_i[\alpha]\mathbf{1}(y_i = y)}{\sum_{i=1}^{n} \mathbf{1}(y_i = y)}$$

$$\sigma^2_{\alpha,y} = \frac{\sum_{i=1}^{n} (x_i[\alpha] - \mu_{\alpha,y})^2\mathbf{1}(y_i = y)}{\sum_{i=1}^{n} \mathbf{1}(y_i = y)}$$

Case study

Formulate the joint conditional distribution

$$P(x \mid y) = \prod_{\alpha=1}^{d} P(x[\alpha] \mid y)$$



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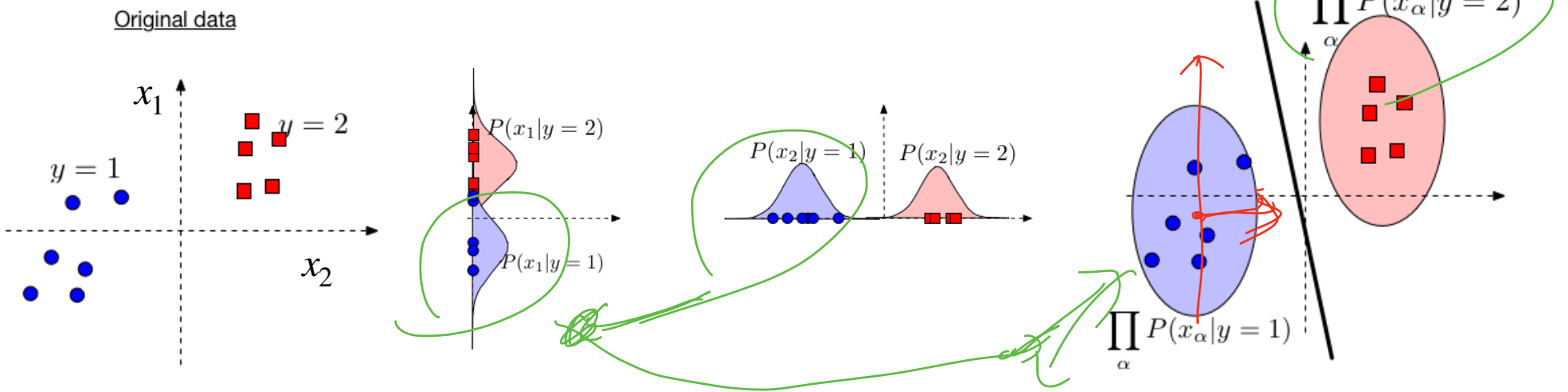


Outline

1. General formulation of Naive Bayes 

2. Example 

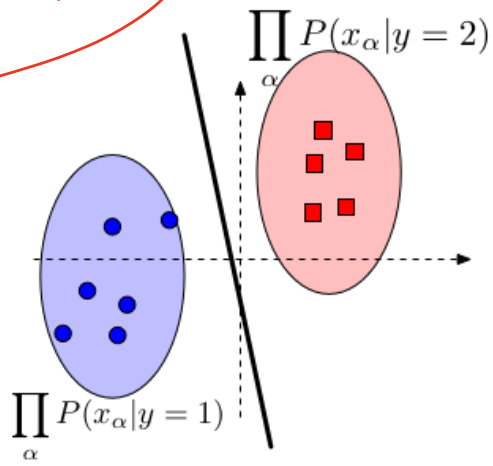
3. Connection to linear classifier

Gaussian Naive Bayes induces a linear classifier

When $P(x[\alpha] \mid y) = \mathcal{N}(\mu_{\alpha,y}, \sigma_\alpha^2)$

i.e., give α, STD σ_α is the ~~same across all~~ labels,

$$\operatorname*{argmax}_y P(y \mid x)$$



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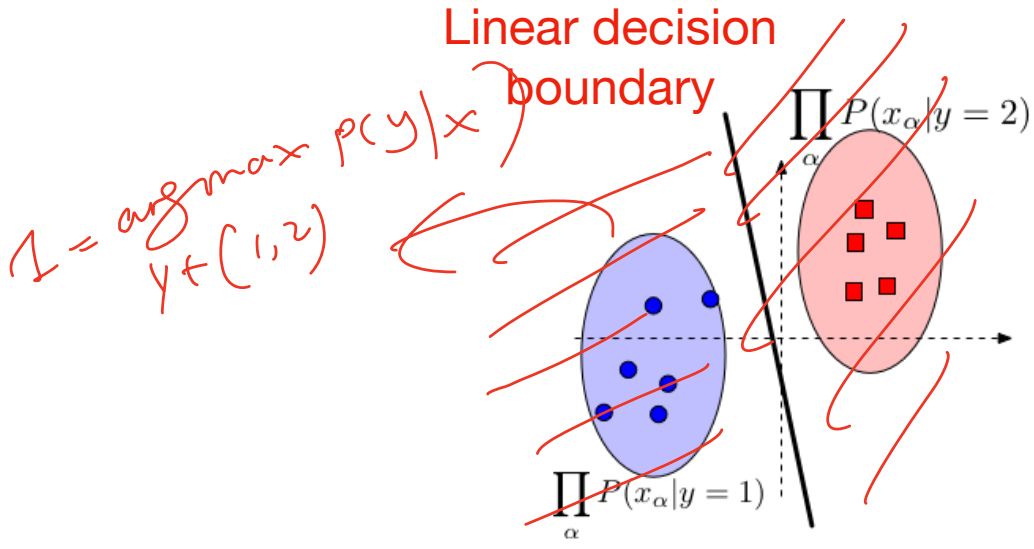

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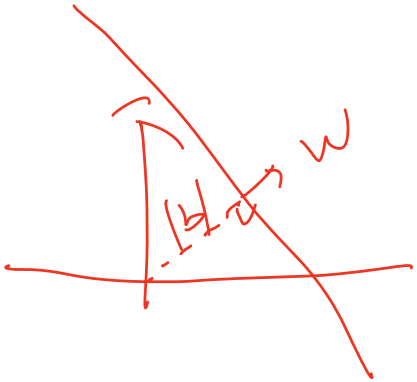
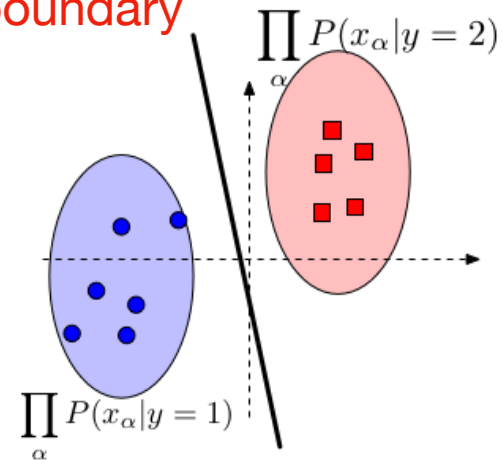


When $P(x[\alpha] \mid y) = \mathcal{N}(\mu_{\alpha,y}, \sigma_\alpha^2)$

i.e., give α, STD σ_α is the same across all labels,

$\exists w, b$ (i.e., a hyperplane), such that:

Linear decision boundary



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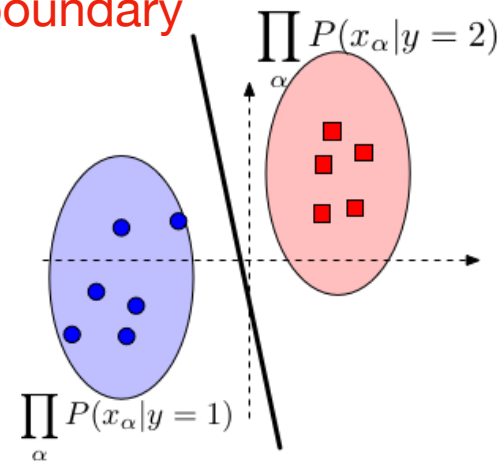

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$$\arg\max_y P(y \mid x) = 1 \iff w^\top x + b > 0$$

Gaussian Naive Bayes induces a linear classifier

When $P(x[\alpha] \mid y) = \mathcal{N}(\mu_{\alpha,y}, \sigma_\alpha^2)$

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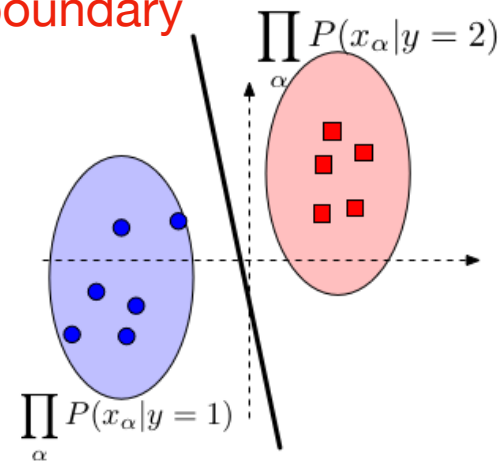

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(Try this out in HW3)

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Take-home Q: Perceptron VS NB classifier