# Boosting

# Announcements

# Recap on Bagging

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$$\hat{h} = \sum_{i=1}^{m} \hat{h}_i / m$$

# Outline of Today

1. Gradient Descent without accurate gradient

2. Boosting as Approximate Gradient Descent

3. Example: the AdaBoost Algorithm

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When $\eta$ is small and $g_t \neq 0$, we know $L(y_{t+1}) < L(y_t)$

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Q: Under what condition of $\hat{g}_t$, can we still guarantee $L(y_{t+1}) < L(y_t)$?

A: As long as $\left\langle \hat{g}_t, \nabla L(y_t) \right\rangle > 0$

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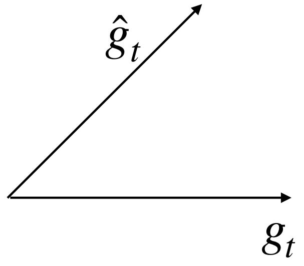

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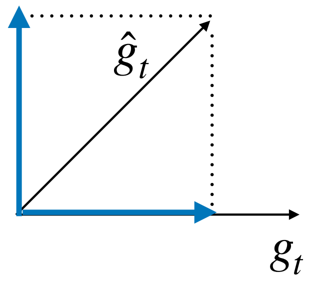

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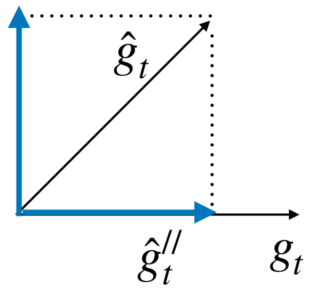

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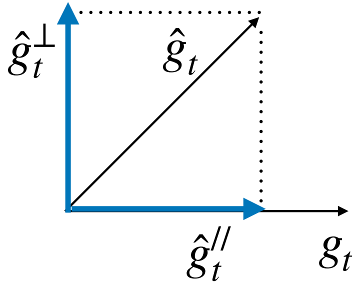

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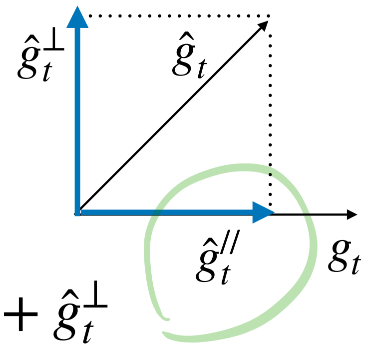


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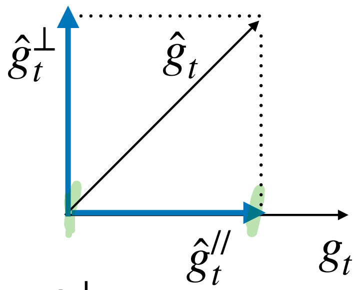


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$:= \alpha$

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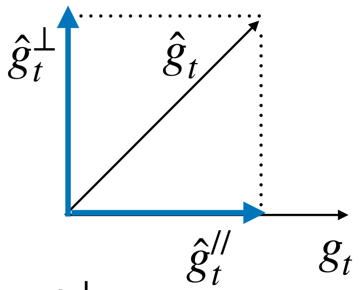
Prove this via first order Taylor expansion and the fact that $\hat{g}_t^\top g_t > 0$

$$L(y_{t+1}) = L(\underbrace{y_t}_{x} - \eta \hat{g}_t)$$

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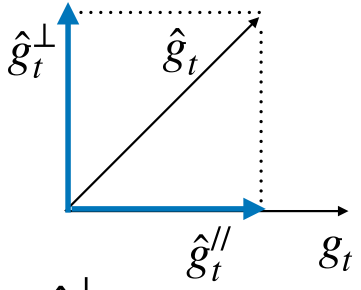


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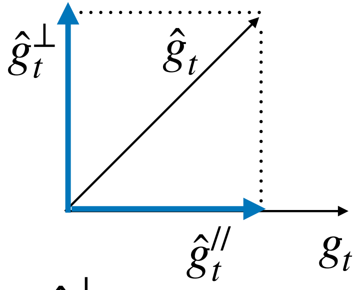


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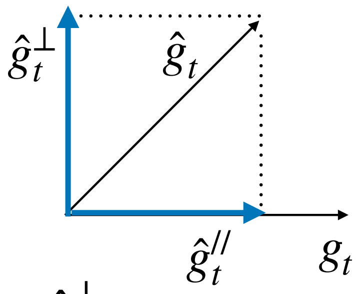


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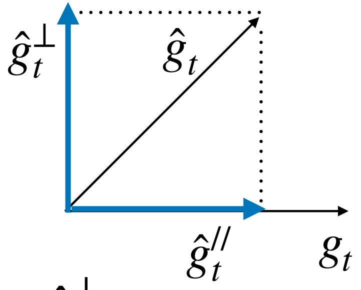
Positive since $\alpha > 0$

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# Key question that Boosting answers:

Can weak learners be combined together to generate a strong learner with low bias?

(Weak learners: classifiers whose accuracy is slightly above 50%)

# Setup

We have a binary classification data $\mathcal{D} = \{x_i, y_i\}_{i=1}^{n}, (x_i, y_i) \sim P$

Hypothesis class $\mathcal{H}$, hypothesis $h : X \mapsto \{-1, +1\}$

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Goal: learn an ensemble $H(x) = \sum_{t=1}^{T} \alpha_t h_t(x)$, where $h_t \in \mathscr{H}$

# The Boosting Algorithm

Initialize $H_1 = h_1 \in \mathscr{H}$

For t = 1 …

Find a new classifier $h_{t+1}$, s.t., $H_{t+1} = H_t + \alpha h_{t+1}$ has smaller training error

$y_{t+1}$ $y_t$ $g_t$

# Training weak learners

$$H_t = \sum_{i=1}^{t} \alpha h_t$$

Denote $\hat{\mathbf{y}} = \left[H_t(x_1), H_t(x_2), \ldots, H_t(x_n)\right]^\top \in \mathbb{R}^n$

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Q: To minimize $L(\hat{\mathbf{y}})$, cannot we just do GD on $\hat{\mathbf{y}}$ directly?

A: no, we want find $\hat{\mathbf{y}}$ that minimizes $L$, but it needs to be from some ensemble $H$

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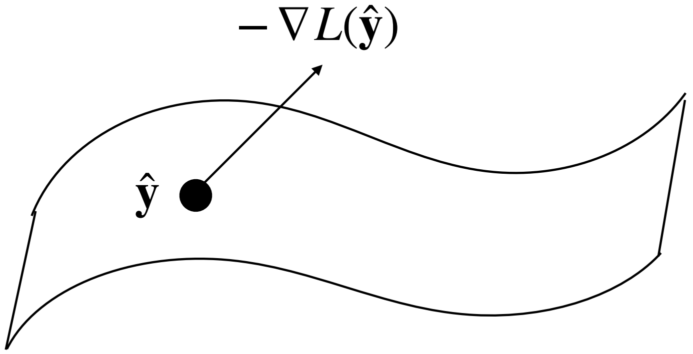
Let us compute $\nabla L(\hat{\mathbf{y}}) \in \mathbb{R}^n$ — the ideal descent direction



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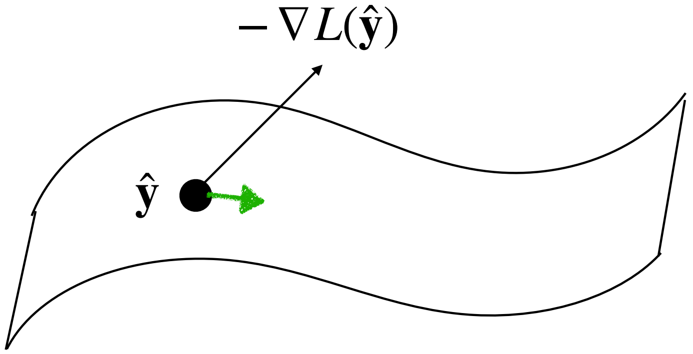

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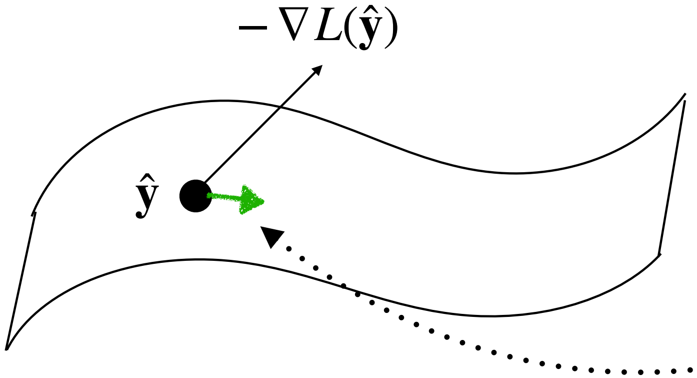
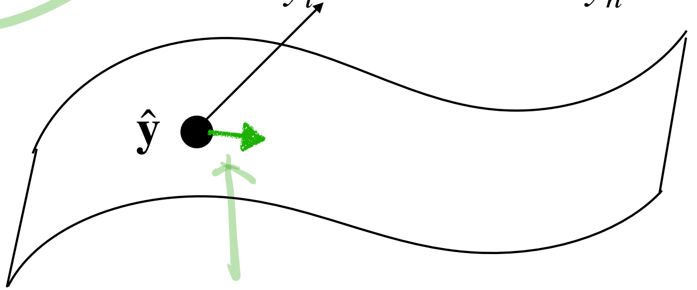



Idea: find a $h \in \mathscr{H}$, such that $[h(x_1), \ldots h(x_n)]^\top$ is close to $-\nabla L(\hat{\mathbf{y}})$

# Training weak learners

$$\hat{y}_i = H_t(x_i) \quad \in \mathbb{R}^n$$

$$-\nabla L(\hat{\mathbf{y}}) = [-\frac{\partial \ell(\hat{y}_i, y_i)}{\partial \hat{y}_i}, \ldots, -\frac{\partial \ell(\hat{y}_n, y_n)}{\partial \hat{y}_n}]^\top$$




$$[h(x_1), \ldots, h(x_n)]^T \in \mathbb{R}^n$$

$$\max_{h \in \mathcal{H}} \begin{bmatrix} h(x_1) \\ h(x_2) \\ \vdots \\ h(x_n) \end{bmatrix}^T -\nabla L(\hat{y})$$

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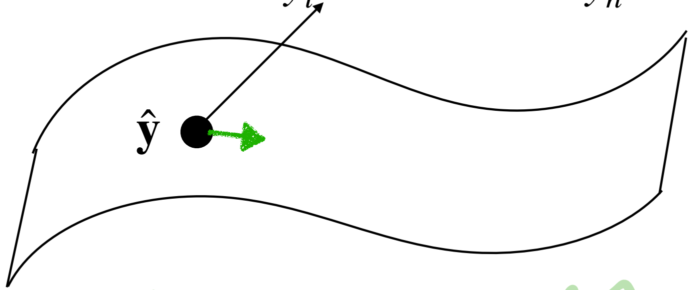
$$\arg\min_{h \in \mathcal{H}} \sum_{i=1}^{n} h(x_i) \cdot \underbrace{\frac{\partial \ell(\hat{y}_i, y_i)}{\partial \hat{y}_i}}_{:= w_i}$$




$$w_i = |w_i| \cdot \text{sign}(w_i)$$

$$\max \sum_{i=1}^{n} h(x_i)\left(-\frac{\partial \ell(\hat{y}_i, y_i)}{\partial \hat{y}_i}\right)$$

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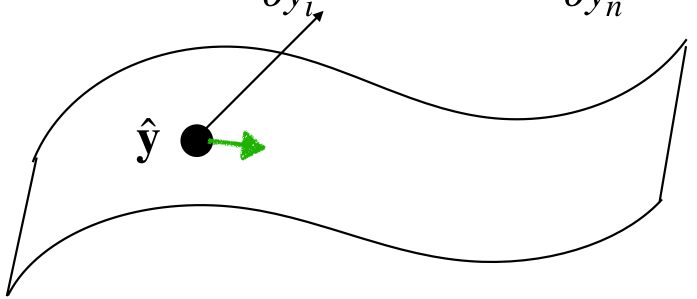


$$\arg\min_{h\in\mathcal{H}} \sum_{i=1}^{n} h(x_i) \cdot \underbrace{\frac{\partial \ell(\hat{y}_i, y_i)}{\partial \hat{y}_i}}_{:=w_i}$$

$$= \arg\min_{h\in\mathcal{H}} \sum_{i=1}^{n} |w_i| \underbrace{\left(h(x_i) \cdot \text{sign}(w_i)\right)}$$

$$= \begin{cases} 1 & \text{if } h(x_i) = \text{sign}(w_i) \\ -1 & \text{else} \end{cases}$$

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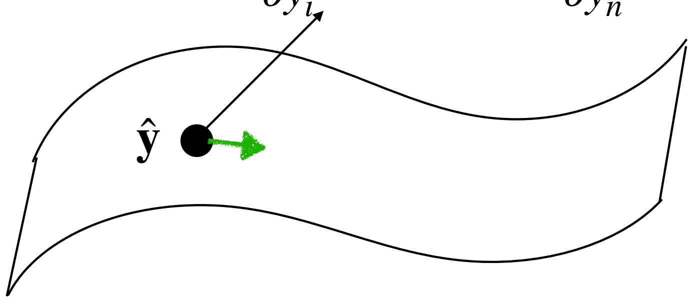


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$$= \arg\min_{h \in \mathcal{H}} \sum_{i=1}^{n} |w_i| \left(h(x_i) \cdot \mathrm{sign}(w_i)\right)$$

$$= 1 - 1(h(x_i) = \mathrm{sign}(w_i))$$

$$= \arg\min_{h \in \mathcal{H}} \sum_{i=1}^{n} |w_i| \left(\mathbf{1}(h(x_i) = \mathrm{sign}(w_i)) - \mathbf{1}(h(x_i) \neq \mathrm{sign}(w_i))\right)$$

$$= 2\, 1(h(x_i) = \mathrm{sign}(w_i)) - 1$$

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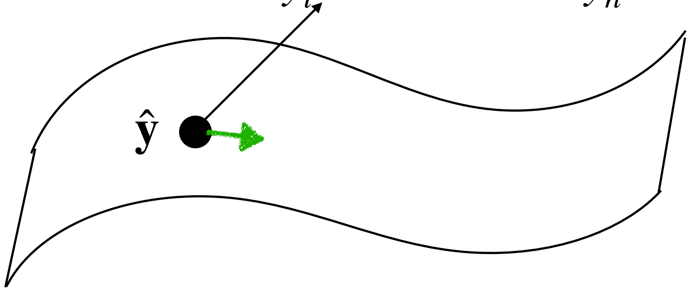


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$= \mathbf{1}\left(h(x_i) \neq -\mathrm{sign}(w_i)\right)$

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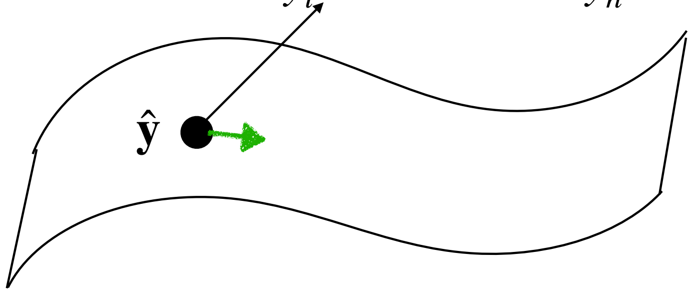


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$$= \arg\min_{h \in \mathcal{H}} \sum_{i=1}^{n} |w_i| \cdot \mathbf{1}(h(x_i) = \mathrm{sign}(w_i)) \quad = \arg\min_{h \in \mathcal{H}} \sum_{i=1}^{n} |w_i| \cdot \mathbf{1}(h(x_i) \neq -\mathrm{sign}(w_i))$$

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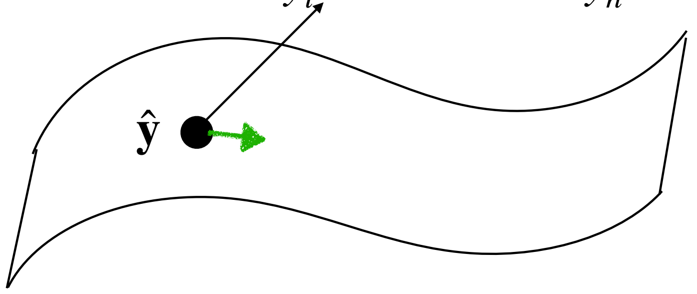
$$-\nabla L(\hat{\mathbf{y}}) = [-\frac{\partial \ell(\hat{y}_i, y_i)}{\partial \hat{y}_i}, \ldots, -\frac{\partial \ell(\hat{y}_n, y_n)}{\partial \hat{y}_n}]^\top$$




$$\arg\min_{h \in \mathscr{H}} \sum_{i=1}^{n} h(x_i) \cdot \underbrace{\frac{\partial \ell(\hat{y}_i, y_i)}{\partial \hat{y}_i}}_{:=w_i}$$

$$= \arg\min_{h \in \mathscr{H}} \sum_{i=1}^{n} |w_i| \left(h(x_i) \cdot \text{sign}(w_i)\right)$$

$$= \arg\min_{h \in \mathscr{H}} \sum_{i=1}^{n} |w_i| \left(\mathbf{1}(h(x_i) = \text{sign}(w_i)) - \mathbf{1}(h(x_i) \neq \text{sign}(w_i))\right)$$

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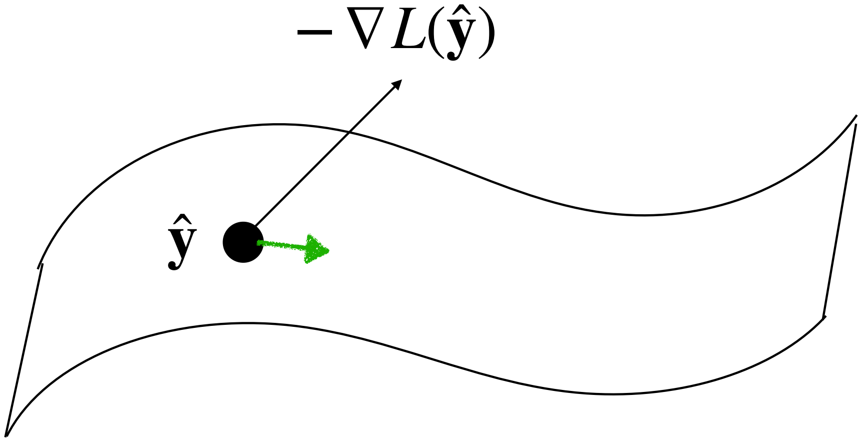
$(x_i, y_i = -\text{sign}(w_i))$

Turned it to a weighted classification problem!

# Training weak learners

Finding $[h(x_1), \ldots, h(x_n)]^\top$ that is close to $-\nabla L(\hat{\mathbf{y}})$ can be done via weighted binary classification:



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A new training set:

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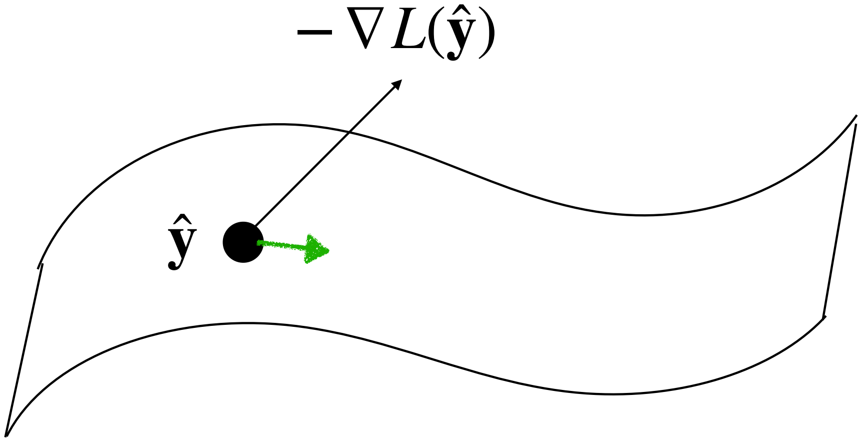

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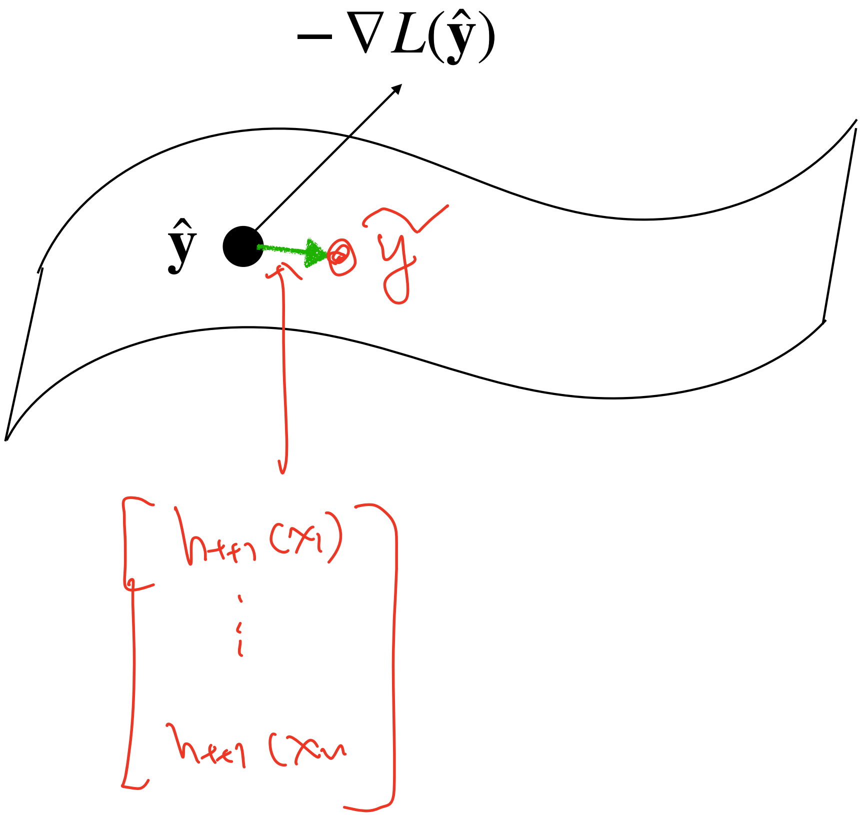

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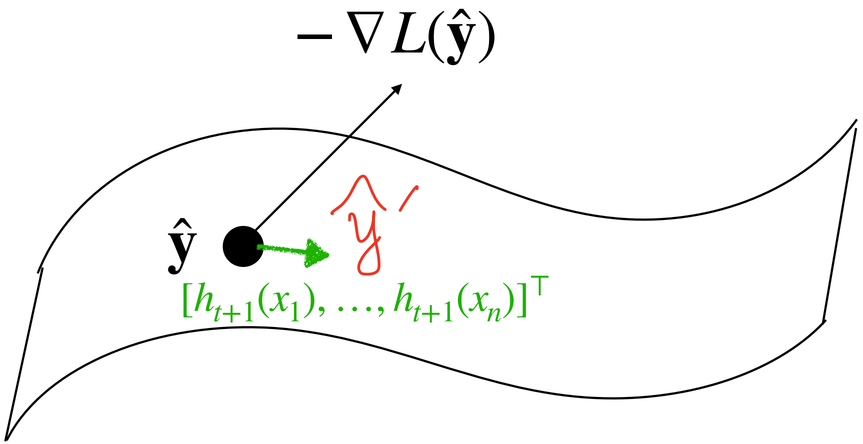

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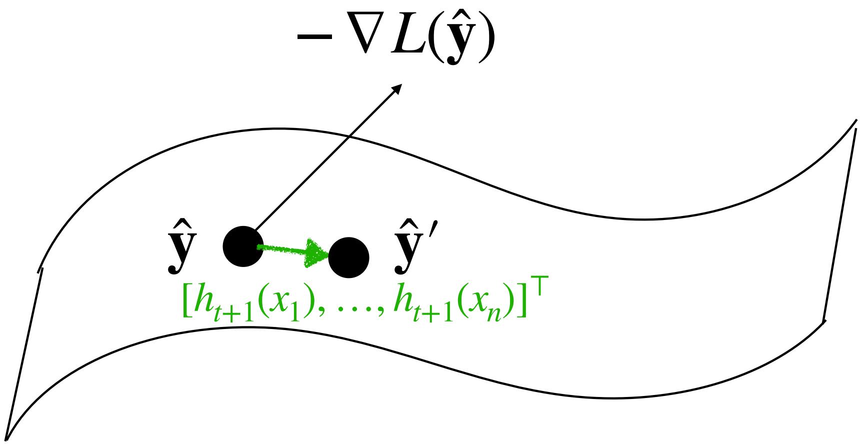

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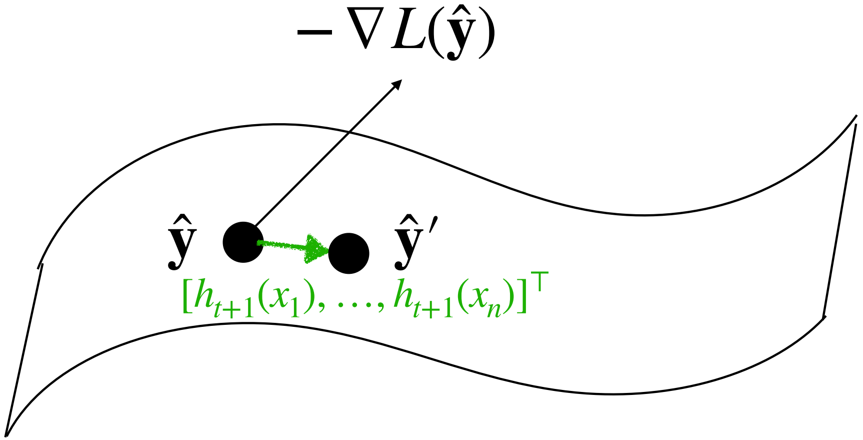


$$\hat{\mathbf{y}}' = \hat{\mathbf{y}} + \alpha[h_{t+1}(x_1), \ldots, h_{t+1}(x_n)]^\top$$

$$\hat{y} = \begin{bmatrix} H_t(x_1) \\ \vdots \\ H_t(x_n) \end{bmatrix}$$

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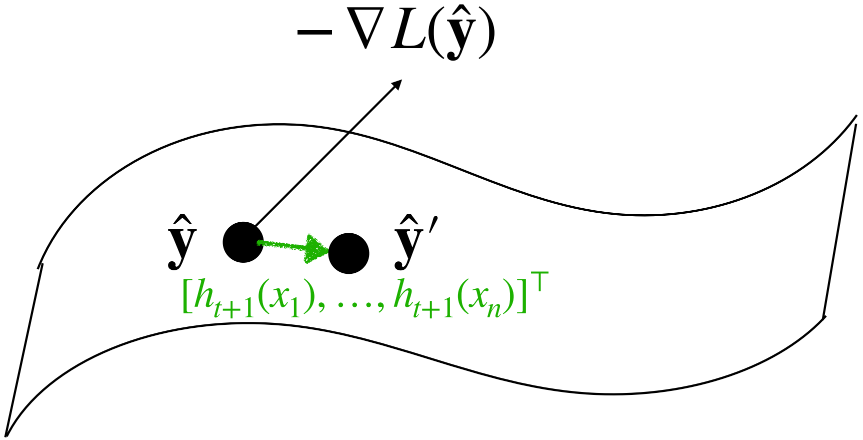


$$\hat{\mathbf{y}}' = \hat{\mathbf{y}} + \alpha[h_{t+1}(x_1), \ldots, h_{t+1}(x_n)]^\top$$

$$= \left[H_t(x_1) + \alpha h_{t+1}(x_1), \ldots, H_t(x_n) + \alpha h_{t+1}(x_n)\right]^\top$$

$$= \left[(H_t + \alpha h_{t+1})(x_1) \ldots (H_t + \alpha h_{t+1})(x_n)\right]$$

# The Boosting Algorithm Revisit

Initialize $H_1 = h_1 \in \mathscr{H}$

For t = 1 …

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Compute $w_i := \partial \ell(\hat{y}_i, y_i) / \partial \hat{y}_i$, and normalize $p_i = |w_i| / \sum_j |w_j|, \forall i$

$= \sum_i p_i = 1, \quad p_i \geq 0$

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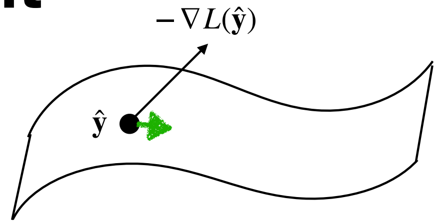


Initialize $H_1 = h_1 \in \mathscr{H}$

For t = 1 …

Compute $\hat{y}_i = H_t(x_i), \forall i \in [n]$

$$\arg\max_{h \in \mathscr{H}} (-\nabla L(\hat{\mathbf{y}}))^\top \begin{bmatrix} h(x_1) \\ h(x_2) \\ \cdots \\ h(x_n) \end{bmatrix}$$

Compute $w_i := \partial \ell(\hat{y}_i, y_i)/\partial \hat{y}_i$, and normalize $p_i = |w_i| / \sum_j |w_j|, \forall i$

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# Outline of Today

1. Gradient Descent without accurate gradient

2. Boosting as Approximate Gradient Descent

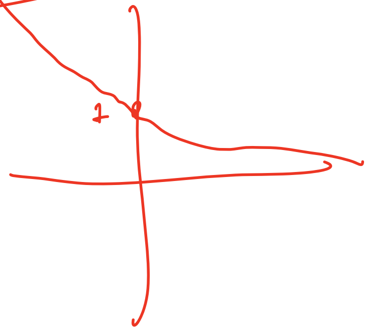
3. Example: the AdaBoost Algorithm

# Train Weak learner

We will choose the exponential loss, i.e., $\ell(\hat{y}, y) = \exp(-y \cdot \hat{y})$

$$\max_h \sum_{i=1}^{n} h(x_i) \left( - \frac{\partial \ell(\hat{y}_i, y_i)}{\partial \hat{y}_i} \right)$$

$$\frac{\partial \ell(\hat{y}_i, y_i)}{\partial \hat{y}_i} = -y_i \exp(-y_i \hat{y}_i)$$



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$$w_i = \partial \ell(\hat{y}_i, y_i)/\partial \hat{y}_i = -\exp(\hat{y}_i y_i) y_i$$

$$p_i = \frac{|w_i|}{\sum_{i=1}^{n} |w_i|}$$

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$$\text{sign}(w_i) = \text{sign}(-\exp(\hat{y}_i y_i) y_i)$$

$$= -\text{sign}(\exp(\hat{y}_i y_i) y_i) = -\text{sign}(y_i)$$

$>0$

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Binary classification on weighted data

$\widetilde{\mathcal{D}} = \{p_i, x_i, y_i\}$, where $\sum_i p_i = 1, p_i \geq 0, \forall i$

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Binary classification on weighted data

$$\widetilde{\mathcal{D}} = \{p_i, x_i, y_i\}, \text{ where } \sum_i p_i = 1, p_i \geq 0, \forall i$$

Q: what does it mean if $p_i$ is large?

$p_i \propto |w_i| = \exp(-\hat{y}_i \cdot y_i)$

# Compute learning rate

Select the best learning rate $\alpha$

$$h_{t+1} = \arg\min_{h \in \mathscr{H}} \sum_{i=1}^{n} p_i \cdot \mathbf{1}(h(x_i) \neq y_i) \qquad H_{t+1} = H_t + \alpha h_{t+1}$$

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Find the best learning rate via optimization:

$$\arg\min_{\alpha > 0} \sum_{i=1}^{n} \ell(H_t(x_i) + \alpha h_{t+1}(x_i), y_i)$$

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Compute the derivative wrt $\alpha$, set it to zero, and solve for $\alpha$

# Put everything together: AdaBoost

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For t = 1 …

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$-\frac{\partial \ell}{\partial \hat{y}_i}$

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$$H_{t+1} = H_t + \frac{1}{2} \ln \frac{1 - \epsilon}{\epsilon} \cdot h_{t+1}$$

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$$H_{t+1} = H_t + \frac{1}{2} \ln \frac{1-\epsilon}{\epsilon} \cdot h_{t+1}$$ // the best $\alpha = 0.5 \ln((1-\epsilon)/\epsilon)$